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THE MATHEMATICS TEACHER

VOLUME XXII JANUARY, 1929 NUMBER 1

THE NEED FOR MORE ADEQUATE MEASUREMENTS OF ACHIEVEMENT IN ARITHMETIC

BY PAUL V. SANGREN

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PROBLEMS OF MEASUREMENT IN ARITHMETIC

Complexity of Abilities in Arithmetic.—In all attempts at measurement the primary problem is that of defining the ability, quality, or factor to be measured. In this respect the abilities in arithmetic offer no exception. When the question is raised as to the exact meaning of that which we call arithmetical ability the most capable student of the psychology of arithmetic must "side-step" with a statement which attempts to state the various directions in which the abilities in arithmetic function. One does not need to be a careful student of Education, Psychology or Mathematics to be aware of the fact that it is impossible to point out a particular mental activity or mental product and say, "This alone is arithmetic or the exercise of arithmetical ability." It will be clear to the most superficial student that in arithmetic will be found the abilities to perform the fundamental operations on many levels of difficulty and to solve verbal problems of many varieties. Any student of mathematics will be aware that the mental tasks involved in working examples or in solving problems in arithmetic may include different degrees of attention, different powers of association and memory, and different qualities of reasoning.

Thorndike [1] has made a serious attempt to outline the general nature of abilities in arithmetic and the directions in which they function, as follows:

[1] Thorndike, E. L., *The Psychology of Arithmetic,* Macmillan Co., Chicago, 1922, pages 23–24.

"(1) Working knowledge of the meanings of numbers as names for certain sized collections, for certain relative magnitudes, the magnitude of unity being known, and for certain centers or nuclei of relations to other numbers.

"(2) Working knowledge of the system of decimal notation.

"(3) Working knowledge of the meanings of addition, subtraction, multiplication, and division.

"(4) Working knowledge of the nature and relations of certain common measures.

"(5) Working ability to add, subtract, multiply, and divide with integers, common and decimal fractions, and denominate numbers, all being real positive numbers.

"(6) Working knowledge of words, symbols, diagrams, and the like as required by life's simpler arithmetical demands or by economical preparation therefor.

"(7) The ability to apply all the above as required by life's simpler arithmetical demands or by economical preparation therefor, including (7*a*) certain specific abilities to solve problems concerning areas of rectangles, volumes of rectangular solids, percents, interest, and certain other common occurrences in household, factory, and business life."

Ruch,[2] Studebaker, Greene, and Knight have attempted to make a more or less complete analysis of abilities in arithmetic in the construction of the Compass Diagnostic Tests as follows:

Test I: Addition of Whole Numbers (Grades 2–8)
Test II: Subtraction of Whole Numbers (Grades 2–8)
Test III: Multiplication of Whole Numbers (Grades 3–8)
Test IV: Division of Whole Numbers (Grades 4–8)
Test V: Addition of Fractions and Mixed Numbers (Grades 5–8)
Test VI: Subtraction of Fractions and Mixed Numbers (Grades 5–8)
Test VII: Multiplication of Fractions and Mixed Numbers (Grades 5–8)
Test VIII: Division of Fractions and Mixed Numbers (Grades 5–8)
Test IX: Addition, Subtraction, and Multiplication of Decimals (Grades 5–8)
Test X: Division of Decimals (Grades 6–8)
Test XI: Addition and Subtraction of Denominate Numbers (Grades 6–8)
Test XII: Multiplication and Division of Denominate Numbers (Grades 6–8)
Test XIII: Mensuration (Grades 7–8)
Test XIV: The Basic Facts of Percentage (Grades 6–8)
Test XV: Interest and Business Forms (Grades 7–8)
Test XVI: Definitions, Rules, and Vocabulary of Arithmetic (Grades 4–8)
Test XVII: Problem Analysis: Elementary (Grades 5–6)
Test XVIII: Problem Analysis: Advanced (Grades 7–8)
Test XIX: General Problem Scale: Elementary (Grades 5–6)
Test XX: General Problem Scale: Advanced (Grades 7–8)

[2] Ruch, Studebaker, Greene, Knight, *Manual of Directions for Compass Diagnostic Test,* Scott, Foresman & Co., Chicago, 1925, pages 9–12.

These two summaries make it sufficiently clear that one of the major problems in the measurement of ability in arithmetic lies in the complexity of the arithmetical processes and in the difficulty of defining the nature of arithmetical abilities.

Ultimate Aims and Goals in Arithmetic.—Another problem which confronts us in attempts at measurement in arithmetic deals with the ultimate aims and goals of instruction. If the ultimate aim of instruction in arithmetic is the development of the general process of reasoning or the disciplining of certain general powers of the mind, the procedure in measurement should be quite different from that employed if the aim is to develop certain specific skills and habits involved in the solution of the practical arithmetical problems of everyday life. It is necessary to be reminded that the comparison of the disciplinary versus the specific aim of instruction in arithmetic in the present discussion is by no means uncalled for inasmuch as a comparatively recent report of the National Committee on Study of Mathematics gives a very considerable portion of its attention to a statement of the disciplinary values in mathematics. Furthermore, it is true that psychologists and educators are to-day studying the entire problem of discipline and transfer of training with greater earnestness and care than was the case a number of years ago when it was supposed that the entire disciplinary theory had finally been "exploded."

It is likewise true that the standards set up for instruction in arithmetic must influence greatly both the extent and type of measurement. If it should be accepted that the ultimate aim of instruction is to develop such skills and cultivate such habits as will enable the child to solve the practical problems of everyday life, the question still remains: How much skill and development are necessary to solve the practical problems of everyday life? Similarly, if we say that pupils should know enough arithmetic so that they can handle the work in high school mathematics with effectiveness, the question remains: How much knowledge of arithmetic is essential to a successful handling of high school mathematics? Since neither of these questions has been answered, the problems of what should be measured and what should be the standards in measurement still puzzle us.

Content of the Curriculum.—A real problem of measurement in arithmetic is concerned with the content of the curriculum. For

several years we have been interested in the type of material which should be included in elementary school arithmetic and the relative emphasis which should be placed on different topics and processes. There has been a great deal of attention given to such topics as: Should only practical and natural problem situations be presented to children? Should problems in plastering, carpentry, and so forth be included? Should square root, cube root, highest common divisor, lowest common multiple, and so forth be included? Ought compound interest, partial payments, discount, and so forth to be eliminated? It is very obvious that measurement is greatly concerned with this matter of content, for any adequate measurement must cover no more and no less than that which should properly be included in the curriculum. Thorndike [3] has said:

> "When we have an adequate sociology of arithmetic, stating accurately just who should use each arithmetical ability and how often, we shall be able to define the task of the elementary school in this respect."

Similarly, it may be said that, when the task of the elementary school has been defined, we may be able to determine what and how much to measure in arithmetic.

Advantages of Measurement in Arithmetic

There are certain decided advantages which measuring devices in arithmetic will hold over measurement of abilities in other school subjects. I shall discuss these advantages in terms commonly used to characterize all educational tests, namely: validity, reliability, and objectivity.

Validity.—By validity is meant the degree to which a test measures that which it purports to measure. Thus, in arithmetic tests the question is: Does the test really measure ability in arithmetic? There are several reasons for believing that tests in arithmetic possess greater valdity than many educational tests. According to any ordinary interpretation few people would doubt that problems like

$2 + 3 = ?$ $\quad\quad$ $5 \times 6 = ?$ $\quad\quad$ $72 \div 9 = ?$

Nell is 13 years old, Mary is 9 years old. How much younger is Mary than Nell?

[3] Thorndike, E. L., *The Psychology of Arithmetic*, Macmillan Co., Chicago, 1922, page 26.

involve primarily abilities in arithmetic. On the other hand, there might be considerable doubt as to whether reading ability is the primary ability measured in the test item:

> Mary said the other day that when she was old enough she was going to be a school teacher. But Mary is only ten years old. She will not be able to teach school for eight years yet. She will then be eighteen years old. Draw a circle around the figure right below to tell how many years it will be before Mary can teach school.
>
> 9 18 8 10 3

The fact that there is more or less direct relationship between specific bonds in arithmetic makes for greater validity in measurement. For example, there is considerable likelihood that pupils who know the combination $3 + 2$ will know the combination $2 + 3$ as well as $2 + 1$, $1 + 2$, $2 + 2$, and so forth. This is not so true in other fields. The fact that a child spells "dog" correctly gives no assurance that he will spell "cat" correctly.

Greater validity in arithmetic tests is made possible because of the more frequent use of the recall type of response in measuring the abilities. The solution of practically all arithmetic problems makes it possible and necessary for the individual to construct his own response. In many other tests it has been necessary in the interest of economy to suggest alternate or multiple responses to test items, thus introducing a factor which tends to reduce the validity of the measurement itself.

Reliability.—Reliability is the degree to which a test measures the same ability at different times or the degree of consistency of the test. Thus, in an arithmetic test, the question would be: Does the child get the same relative score on the test at different times? There are two reasons why somewhat greater reliability may be expected in arithmetic tests than in certain other educational tests. First of all, it is possible to multiply to great length arithmetic problems which are approximately equal in difficulty and involve the same processes or problems which increase in difficulty by fairly even steps and involve higher and higher mental processes. In such subjects as reading, for example, this would not be true because no two reading situations, unless decidedly artificial, could be exactly alike in difficulty and process involved. Secondly, varying inferences, opinions and interpretations cannot so much influence the response in arithmetic problems. On the other hand in reading, for example, the response

to a particular reading situation is greatly influenced by such factors as vocabulary, home training, emotional setting, and reading between the lines.

Objectivity.—Objectivity is the degree to which a test is free from the personal opinion and training of the examiner. Arithmetic tests may naturally be expected to be more objective. Responses to arithmetic problems are stated in the most exact and objective terms known to man. There is usually little debate concerning the correct answer to an arithmetic problem. Furthermore, in the administration of the test the examiner exercises a minimum of influence on the pupil being tested because the test is usually set up in a way that is common to the school situation, enabling the child to recognize at once the nature of the task before him. Neither of the two facts just stated will hold so generally in the measurement of abilities in other school subjects.

Existing Tests and Scales in Arithmetic

Original Standardized Tests.—In the original construction of standardized tests in the field of arithmetic test authors used the traditional divisions in the subject matter of arithmetic, namely, fundamental operations and problem solving. The writer may briefly describe certain of these tests as they exist to-day. In measuring ability in the fundamental operations, the Courtis Standard Research Tests in Arithmetic are typical of tests measuring skill or efficiency. As you know, the Courtis Tests are constructed in such a way that separate rate and accuracy scores may be obtained by working the examples in addition, subtraction, multiplication and division on a single level of difficulty. These tests are adjusted to such a level of difficulty that pupils in grades IV to VIII may be expected to handle them with different degrees of efficiency. The Woody Arithmetic Scales are typical of tests measuring development. All are familiar with the fact that these tests are so constructed that the examples run from easy to difficult and give separate scores for addition, subtraction, multiplication, and division, indicating, in general, how difficult an example in each operation a child can perform. The tests are used in grades IV to VIII. In measuring efficiency in problem solving the Monroe Standardized Reasoning Test in Arithmetic is typical. This test is constructed in such a way

that problems of approximately the same degree of difficulty are placed before the pupil and blank space is provided for him to work out the problems in detail. The pupil is scored both for correct solution and correct principle. The following example is taken from the test:

8. At 3 cents per foot, what is the cost of sufficient picture molding to go around a room 14 ft. by 14 ft.?

The test may be applied in grades IV to VIII. For measuring development in problem solving the Arithmetic Reasoning Test of the Stanford Achievement series is typical. This test is so constructed that problems, running from easy to hard, are placed before the pupil and he is asked to find the correct answers. The test determines how difficult a problem the pupil can perform. The following are samples of the easiest and most difficult problems:

1. How many are 3 eggs and 2 eggs?

40. If the hour hand of a clock is 3 inches long and the minute hand is 4 inches long, how far apart are the tips of the two hands at 9 A.M.?

The test may be applied in grades IV to VIII.

These four tests, the Courtis, Woody, Monroe, and Stanford, are very useful from the standpoint of determining the relative efficiency and development of pupils in the fundamental operations and problem solving in arithmetic. Speaking in analogy, these tests may tell that the individual is ill but they give no indication of the nature of the disease. Recently the writer placed before a group of teachers the following facts obtained from application of the Courtis Standard Research Tests in Addition to a group of fourth grade pupils:

	Rate	Accuracy
School A Grade IV	6.4	72
Courtis Norm	7.4	62

These facts were accompanied by the question: "What should be done on the basis of these facts?" Some of the replies follow:

1. "Give the pupils speed drills on the type of examples included in the Courtis Test."

2. "Give the pupils speed drills on simple addition combinations first and gradually increase their complexity until they are on the level of the examples included in the Courtis Test."

3. "Use the Courtis Practice Exercises in Addition."
4. "Determine which individual pupils are deficient in rate of addition and give them speed drills."
5. "Determine the causes for the deficiency of the group in rate of addition by the use of some diagnostic tests."
6. "Pay no attention to the rate of the pupils' addition. It is of minor importance."

This is an example of the type of interpretation and probable procedure which will follow the application of some of these original tests in arithmetic. While they are very helpful in determining the status of pupils, they give no adequate description of their particular strengths and weaknesses.

Modern Standardized Tests.—The most recent standardized tests in arithmetic are much more analytical and diagnostic in character than the original tests which have just been discussed. The writer will attempt to describe the nature of these measuring devices as briefly as possible.

The Wisconsin Inventory Tests in Arithmetic typify a certain tendency toward a more analytical and diagnostic measurement of abilities in the fundamental operations. This series of tests confines itself to the measurement of skill or efficiency in the fundamental operations according to the following plan:

Test I includes the 100 combinations in first decade addition with such examples as

$$\begin{array}{ccc} 0 & 5 & 0 \\ 2 & 7 & 3 \\ \hline \end{array}$$

Test II includes the 100 combinations in first decade subtraction with such examples as

$$\begin{array}{ccc} 8 & 12 & 16 \\ 0 & 8 & 7 \\ \hline \end{array}$$

Test III includes the 100 combinations of simple multiplication with such examples as

$$\begin{array}{ccc} 6 & 0 & 5 \\ 1 & 2 & 8 \\ \hline \end{array}$$

Test IV includes the most difficult of the combinations which are needed in short division with such examples as

9)32 7)62 6)40

Test V includes the most useful combinations involved in column addition not covered in Test I with such examples as: 33 and 5 are? 13 and 6 are? 31 and 7 are?

Test VI includes the addition combinations needed for accurate carrying in multiplication with such examples as: 25 and 3 are? 72 and 7 are? 10 and 2 are?

Test VII includes all combinations which give zero in the quotient with such examples as

9)54270 8)8401 9)637

Test VIII includes all the major difficulties as long division with such examples as

352)2140392 312)279864 87)69252

These tests, it will be noted, aim to make a complete inventory of the basic knowledge and skill which the child should possess in order to perform the fundamental operations in arithmetic effectively. These tests contain such injunctions as "Do not teach what the child knows. Teach what he does not know. Use the Wisconsin Inventory Test No. 2 to find where the trouble is in subtraction. Do not teach long division to a child who cannot multiply and subtract." These tests propose to show what specific and underlying weaknesses a pupil has which prevent him from attaining the greatest success in the work of addition, subtraction, multiplication, and division.

The Compass Diagnostic Tests in Arithmetic, to which reference was made in the early part of this paper, typify another tendency in present day measurement of abilities in arithmetic. As you recall, these tests include a series of 20 different measures of the abilities in arithmetic. Consider the measurement of ability in the fundamental operations in arithmetic. The following is a sample of the type of analysis which is made by the Compass Diagnostic series:

1. Ability to add functions differently in the work with:

   Whole numbers
   Fractions and mixed numbers
   Decimals
   Denominate numbers

2. The ability to add whole numbers may function differently in the following cases:

   a. Basic addition facts such as

0 + 2 =	1 + 7 =	0	7
		2	1
		—	—

   b. Higher decade addition such as

2	1	33	1
20	21	3	24
—	—	—	—

c. Column addition such as

4	6	7
7	6	6
5	9	7
—	3	4
	4	8
	—	7
		6
		—

d. Carrying in column addition such as

29	13	117
47	17	960
—	67	517
	—	—

e. Checking answers in addition such as

165	213
923	128
928	456
466	—
—	787
2484	

A similar analysis is made of addition of fractions and mixed numbers, decimals, and denominate numbers. Carry this analysis forward to cover, similarly, the fundamental operations of subtraction, multiplication, and division and the various phases of mensuration, percentage, interest, vocabulary and rules, problem analysis and problem solving and one has a fair notion of the completeness of the series of tests. The authors of the tests state the theory underlying their construction as follows: "Every important ability or skill in arithmetic can be analyzed into a number of simpler constituent skills. Lack of mastery of any one of these means lack of mastery of the total skill. It has often been said that a chain is no stronger than its weakest link. This is particularly true of a highly interrelated subject like arithmetic in which each successive level of complication must be mastered before proceeding to the next. It is not enough to know that a pupil is weak or inaccurate in arithmetic in toto, or in fifth-grade arithmetic in general, or even in long division or decimals taken as a whole. Before the remedy can be found it is necessary to diagnose the specific abilities and skills which are improperly mastered. Until this diagnosis is made, corrective or remedial teaching must remain largely a hit-or-miss proposition."

These two series of tests—the Wisconsin and the Compass—typify the direction of the measurement movement to-day as it relates to arithmetic. It will be clear that there is a very definite attempt not only at determining whether the pupil is arithmetically ill but also at determining what are the nature and causes of his deficiencies.

Need for More Adequate Measures

From the previous discussion it will be clear that many improvements have been made in the measurement of abilities in arithmetic during the past fifteen or twenty years. It will also be clear that there are several directions in which the measurements might be changed and supplemented so as to be more useful for the improvement of instruction in arithmetic. The writer wishes to suggest certain changes and supplements to present methods of measurement in arithmetic relative to the types of tests to be applied and the purposes for their application.

Construction of New Types of Tests.—One of the most obvious needs of measurement in arithmetic to-day is the construction of an *original inventory* test which will determine what sort of arithmetical equipment the child possesses as formal instruction in arithmetic is begun. To date practically all measurement has been made to determine the effects of specific and formal instruction. It is common knowledge that children collect a good deal of mathematical information and ability in the home and at play, all of which gives them varying degrees of preparedness for formal instruction. At the present time this formal instruction in arithmetic begins anywhere from the kindergarten to the fourth grade, depending upon the opinions of those in charge of the curriculum. Would it not be well to construct original inventory tests by means of which it could be determined what abilities in arithmetic a child possesses in his early years and to fix a time for the beginning of formal instruction?

There is also great need at the present time for certain *prognostic* tests in arithmetic. There should be some means whereby one could predict fairly accurately the course of development of the child's abilities in arithmetic. We find that children vary greatly in the readiness with which they absorb the instruction. Some seem to be natural mathematicians; others find arithmetic

a source of greatest unhappiness. It should be possible to construct tests which would anticipate the nature of the difficulties of pupils in arithmetic so that adjustments could be made to avoid the pitfalls of traditional and mass instruction.

In the present stage of the psychology and method in arithmetic there is a definite demand for *instructional* tests to accompany certain textbook series and methods of teaching. It is becoming increasingly evident to us that there should be frequent standardized checks covering certain units of subject matter as treated in the different systems of arithmetic teaching. These instructional tests should serve as a source of continuous study of the growth and development of the pupil, avoiding the present difficulty of measuring the effects of teaching after it is too late to introduce a change or remedy.

There is now occasion for renewed interest in the *practice* tests and exercises which received considerable emphasis ten years ago. All will recall that a number of years ago Courtis constructed a series of standard practice tests in arithmetic which received considerable attention and use. There seems to have been a gradual departure from the employment of this scheme primarily because of the nature of the practice tests themselves. At the time of their construction the Courtis Practice Tests gave emphasis to drill according to the best known psychology of arithmetic. Marked changes have come about in our knowledge of the psychology of arithmetic, the distribution of practice, and the content of the curriculum. To-day we stand in need of the construction of new series of practice tests which will accord with the best known psychology of arithmetic and aims of instruction.

More Effective Application.—Teachers of arithmetic should, first of all, be made aware of the fact that arithmetic as a subject and as an ability is of such a nature that very frequent measurements are justifiable and necessary. Abilities involved in arithmetic are so complex and yet so definitely influenced by instruction that continuous checks should be made upon the progress of pupils. When the argument is advanced that the teacher cannot spend all of her time in giving and scoring tests, the answer is that for years practically all instruction in arithmetic has been testing the pupils. Pupils have been given problems at the board and in their seats; day after day assignments have been

made and the teacher has measured the effectiveness with which the pupils mastered the assignments by traditional and informal measurements. Why not maintain a systematic and standardized method of continuous measurement and recording of the progress of the pupils?

Teachers of arithmetic ought to make more use of standardized measurements as a means of solving their problems of instructional method. Arithmetic has been handed down from generation to generation. Although one cannot justly say that there has been no careful study made of the problems of method in arithmetic, it is certain that no such freedom for study and experimentation has existed among arithmetic teachers as is found among teachers of such a fundamental subject as reading. The time has come when teachers in the schoolroom need to make serious attempts at the comparative evaluation of different methods of instruction in arithmetic by means of carefully tested original surveys and final appraisals both for certain phases of the subject matter and for the course as a whole.

Measurements ought to be applied both in and out of the school for the purpose of more adequately determining the relative values of different phases of arithmetic and the ultimate goals to be attained. As has been previously stated, we stand in such a position to-day that we do not know definitely what is the task of the elementary school as it relates to arithmetic. Only through the application of measurements to pupils in the schools as they go up through the grades, on in to the high school, and as they occupy various industrial and social positions in the world can we definitely determine the relative values to be attached to various phases of arithmetic instruction. Only as we measure the relationship between the school instruction in arithmetic and the life outside can we settle upon our purposes and goals. These matters ought to demand the attention of every conscientious teacher of arithmetic.

A REPLY TO "THE POSITION OF THE HIGH SCHOOL TEACHER OF MATHEMATICS"

By ELIZABETH B. COWLEY
Pittsburgh, Pa.

The article on the above subject in the October number of this journal presents one point of view quite forcibly. But is that the only point of view? I think that it is not. I wish to present another aspect of this matter by commenting upon certain sentences of the above article which appear in italics below. But before doing so it may be well to indicate my academic status. I have my doctorate in Mathematics. After many consecutive years of teaching calculus and various other branches of mathematics in a large eastern college, I am, by invitation, spending my accumulated years of leave of absence doing research work on the mathematics curriculum of the senior high schools of a large city, and at the same time carrying the full work of a regular teacher of mathematics in a coeducational senior high school. I teach nothing but plane and solid geometry.

Is there any doubt that the high school teacher's position is unsatisfactory in general, and that of the mathematics teacher in particular?

Yes, there is grave doubt. There probably are isolated instances here and there, especially in small high schools. But are the high schools any worse than some colleges in their treatment of teachers? It is not necessary for me to answer this question, for everyone who reads this article can cite instances. Even the daily newspapers furnish illustrations.

In some ways the situation in the primary and grade schools is preferable. . . . But in the high school . . . the contact with parent and pupil is not so close.

Did the writer ever hear of the "Home Room," or "Report Room," where the pupil spends all the school day except when in the classes of other teachers? The report teacher receives and records the marks of each of his pupils for each of the three report periods each semester. To him the pupil must present

every excuse for absence, tardiness, early dismissal, or "black list." He guides the "activities" of his pupils and acts as counselor and friend. In short, he stands "in loco parentis" during school hours. I feel that I have a vital relation to the thirty-three boys and six girls in my report room.

The teacher is an insignificant part of the process, a cog in the machine, a link perhaps between the superintendent and the magnificent school buildings, but less important than either.

If this is the situation in any particular case, may the fault not lie with the teacher himself? Probably we have all seen persons sitting in teachers' chairs (both in colleges and in high schools) whom we considered quite insignificant. The institution is blameworthy because it allows this insignificant person to remain upon its staff. Incidentally, not all high school buildings are "magnificent." The one in which I am teaching is forty years old. It is not "up-to-date" as a building, but it has traditions and alumni of whom it is proud.

In the first place, students are made to feel that education is play, not work.

Is there anything to be gained by making the process of education distasteful drudgery? Is a teacher to be criticized if he and his pupils really enjoy the class periods? Was the teacher to be blamed when one of her boys exclaimed "Oh, I got a great kick out of that," when he had finally solved a difficult original in geometry with which he had wrestled for several days? It must not be inferred that pupils who attack geometry as they would a game are poor students. Another boy in the same class has recently won a fellowship at the leading school of electrical engineering (in another city).

The curriculum has been filled with trivial excitements in order to induce the pupil to forget temporarily that his main business is football and baseball.

I wonder whether the writer of that sentence ever taught members of the football squad? I have (or have had) almost every member of our squad and I like them as students. Some of them are among my ablest pupils. All have a sense of honor and fair play that is, to my way of thinking, a most commendable result of any school system. No, it is not the football player that I find troublesome, but that soft, fat fellow who is forever munching huge chocolate creams and who tells false-

hoods and tries to get inspiration from his neighbors' papers, because he is too lazy to work for himself. And, by the way, he is the boy who speaks of lessons as work, not play.

The writer of the article referred to above tells a touching tale of the teacher who thinks that forty percent is "about the right percentage" of failure. Now, honestly, what would you think of a physician or surgeon if forty percent of his patients died? And his position is more difficult than ours; for practically none of his patients are in a normal state of health while the majority of our pupils are healthy and normal. Too many who are paid to teach take the attitude of the college professor who said, "I do my own research work and deliver my lectures. If the students profit, it is well; but if they do not succeed in their work, it is their fault and not mine."

Again, the above article says: "So much for the trifling student." No teacher likes the trifling student. But are the colleges free from students of this type? Many a high school teacher or vocational adviser has urged certain boys and girls who are beyond school age to leave school and go to work. But, according to law, a pupil must be in school until he has reached a certain age. Nor is it to be overlooked that a lad whom one teacher may label a "trifler" may, under the more inspiring leadership of another teacher, develop into the finest type of student. Again, we owe a duty even to the worst type of youth. If he is turned out of school and thus freed from wholesome restraint and the necessity of developing his powers along useful lines, society may pay the price by having another criminal on its hands.

But in the second place the curriculum is now crowded with subjects which are irrelevant to education, like vocational instruction and commercial training. It must not be.

It must be clearly understood that the primary purpose in establishing public high schools is not to furnish an abundant and steady supply of pupils for colleges. Boys and girls differ in aptitude and in interests. Some who could not gain much from an academic course can profit by another type of course. By what authority does a college professor decide upon the question of who shall attend and what the curriculum shall be in the public high schools? They are built and maintained by the taxes of the people—in many cases the parents of the boys and girls who take vocational or commercial courses.

The teachers themselves have not sufficient freedom as individuals. Some of them . . . spend time besides in tabulating all sorts of data.

In the school system in which I am at present, the "data" are chiefly for the State Department of Education Records. All that I have made out are either regarding my own personal academic history or concerning my pupils. I must confess that these things have never been a burden to me. I do them at once. I have always had in my report room a few rapid workers who finish their lessons before the end of the study period and are eager to "help." I have gleaned much information regarding my pupils. For example, a boy who had seemed impenetrably dense came from a Lithuanian home where no English was spoken. Colleges have a fad for statistics too. But they use them for different purposes. I am told that a statistical study of the rural public schools of an eastern state, made by a graduate of a western private school and college, a man who had had no mathematics beyond one year of alegbra won for the person promotion to a full professorship—and it was not in a School of Education.

Finally, the article refers to "the programs that come to him ready made." There must be programs in any large school system. If every teacher taught what he pleased and when he pleased, what would happen to the unfortunate pupil whose parents moved from one part of the city to another? The teachers do have opportunity to express their preference, but obviously it is impossible to please everyone. I have found not only opportunity for "experimenting with" my "own ideas," but I have been encouraged to do so. After all, is it not true that freedom is a state of mind?

Alas, we must admit that neither high schools nor colleges are faultless; both have trifling students, both have men and women who receive teachers' salaries but do not teach. But we must not allow ourselves to be so overwhelmed by these defects that we lose our perspective. Both high schools and colleges have inspiring teachers who are guiding eager, active young people and helping them prepare themselves for the day when they shall step out with the whole world before them.

GENERALIZATION

By ERNEST B. LYTLE
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Generalization as a principle of teaching mathematics is often neglected. We are not thinking here of generalization as simply a process of passing from particulars to generals; rather we wish to consider that attitude of mind which continually gives attention to the relations and significance of facts. Learning is a continuous process, a growth, a development, a passing from one thing to something else. In the process of learning there must be concrete particulars; these particulars are the means to intellectual growth and form the material of which knowledge is made. But so long as they remain just isolated particulars they are of little value, they are mere excitations and possible irritations. To be of value these particulars must be related, organized, given meaning and significance. The mere encyclopædic collection of facts is not our conception of valuable learning; rather it is the interpreting and application of facts, the relation and organization of facts, the careful study of the meaning and significance of particulars which rounds out and carries forward the learning process. This continual seeking for significance and organization is generalization.

To be useful knowledge should be generalized. Helen M. Walker gives [1] a fine illustration of the principle of generalization. The head of an English department was visiting a geometry class to study the language difficulties of the pupils. She heard a lad sum up the results of a demonstration thus: "Therefore if a triangle has AB equal BC, then angle A is equal to angle C." The geometry teacher pointed out to him that information put in so particular a form could not easily be used again; if the knowledge just gained were to serve him again and again in the future he must generalize it; eventually the class worked out the theorem, "If two sides of a triangle are equal, then the angles opposite these sides are equal." Within

[1] THE MATHEMATICS TEACHER, Jan., 1925, p. 50.

a week after this incident this surprising sequel occurred in one of that English teacher's own classes. They were reading Silas Marner and one of the boys offered as his explanation, "Silas was accused of taking money he hadn't stolen, so he just ran away. But it didn't do him any good." At once a girl who had been in the geometry class spoke up saying, "That isn't the way to put it. You can't use this knowledge anywhere else if you make it a special case. You have to say it this way, 'When one is falsely accused of committing a crime, it is useless to try to escape by running away.' "

If vauable learning, then, involves both particulars and generalizations, teaching may fail in two ways. Either there may be too much consideration of unrelated particulars with little or no attention given to significance; or there may be a neglect of particulars and too much repetition without understanding of the generalizations of others. So the successful teacher must pay attention to both particulars and generals keeping a proper balance between them.

The trained mathematician revels in generalizations but is often not understood because he neglects basic and illustrative particulars. Some experience with particulars is necessary to true comprehension of generals, in fact the ability to give particular illustrations of general notions is a test of real comprehension. When your students find textbook statements so general as to be vague, do you find them *naturally* seeking understanding through consciously going back to familiar particulars? My students need continual urging to present illustrative particulars. On the other hand, after discussing, discovering or proving facts do you find your students mentally uneasy and dissatisfied until they have worked out some significance, meaning or application of the new facts? What reaction do you get when you ask your students what they have *learned* in working the problems of an assignment? I find that too many of my students do not meditate enough on their work; they do not naturally seek for significance and need much pushing to get them to make good generalizations. They seem to work problems just for answers, unconcerned over what they have learned or over the meaning of their answer. In practical life, of course, the correct answer is the important thing, but in the learning stage the *significance* of method and result is likewise most important.

In learning algebra, for instance, there should be much meditation on it beyond simply getting answers, for meditation results in necessary generalization which makes the knowledge useful. Students who can work exercises in a text where problems are carefully classified and where model solutions may be followed, often cannot solve miscellaneous problems because they have not generalized methods; many cannot make the common transformations and reductions because they have learned no rules. Mere drill in technique of symbolic manipulation is useless unless final general rules or theorems result. Some teachers want rules printed in the text, but most successful teachers I think prefer that their students make their own rules realizing, however, that students need continually to be directed and urged to generalize their methods and their results.

The author believes that teachers are failing to train students to generalize, else why do students react so poorly when asked for rules of procedure, descriptions of general methods and summaries of chapters, books, or fields of study? The importance of habits of generalization in education has been emphasized in many places in the literature. Judd [2] gives an excellent discussion of generalized experience. Whitehead remarks [3] "We cannot think in terms of an indefinite multiplicity of detail; our evidence can acquire its proper importance only if it comes before us marshalled by general ideas." Dewey says [4] "The *measure of the value* of an experience lies in the perceptions of relationships or continuities to which it leads up"; his discussion of valuable experience is most suggestive to teachers. Dewey in another place [5] says: "Only as general summaries are made from time to time does the mind reach a conclusion or a resting place; and only as conclusions are reached is there an intellectual deposit available in future understanding"; "When a topic is to be clinched so that knowledge of it will carry over into an effective resource in further topics, conscious condensation and summarizing are imperative." Both Dewey and Bode in numerous places have emphasized the great value of general notions rich in suggestive meanings. James story of his struggle

[2] Judd, *Psychology of High School Subjects*, pp. 392–435.

[3] Whitehead, *Atlantic Monthly*, Aug., 1925, p. 204.

[4] Dewey, *Democracy and Education*, p. 164.

[5] Dewey, *How We Think*, pp. 212, 216.

with his smoking lamp and his discussion of generalization [6] are classic.

If habits of generalization are thus so valuable and so important in learning, how may teachers train their students to generalize habitually? One very effective device is to require students, in frequent blackboard work, to give orally the general rule or theorem while simultaneously writing the particulars. The oral statement of general rules or principles as they are used fixes them in the mind and shows students just how steps in the solution of a problem are suggested and directed by general notions previously learned. In giving model solutions textbooks generally tell what operations are performed but almost never tell what suggests the operation used. Students often remark, "I see that the operations performed will solve the problem, but I never thought of doing that; what *suggested* those steps?" Teachers should be careful, in working problems for students, to exhibit all their cues to action; by repeating orally the rules and theorems as they perform the particular operations the students finally catch the idea of how general notions suggest and direct particular acts. We are not contending that rules should *always* be given orally as pencil or crayon work is done; but beginners should do this until the rules and principles are well learned and until they appreciate generalizations as a source of suggestion for particular operations or procedures. Experts often fail as teachers just because they do not exhibit to their students their cues to action, they do not give the suggesting elements in their thinking. I have found that oral expression of general notions, rules and principles as one performs particular operations is an effective method of developing habits of generalization.

The frequent use of the question, "What have you learned?" also helps to form habits of generalization; the act of bringing to consciousness and the precise statement of things learned compels generalization. Continued exercises in summarizing sections or chapters read, helps habits of generalization. Summary analyses of subjects or even whole fields of study are valuable requirements. How long could your students talk on such topics as, "What is arithmetic?", "What is algebra?" To describe and illustrate the typical problems of a field of study in the

[6] James, *Principles of Psychology*, Vol. II, pp. 323–372.

answer to such questions necessitates careful organization and summing up of the things learned. "Four-minute talks" on topics of a course are excellent means to habits of generalization.

A discussion of generalization should of course include a caution about errors of making unwarranted generalizations, and against jumping to wrong conclusions. Vague generalities are too common in both writing and speech, and must be fought by teachers. Care given to precise formulation and continually testing by illustrative particulars, will tend to avoid errors as well as bring about insight and real understanding.

We have seen that both particulars and generals are necessary in the learning process and their proper balance is vital in teaching. Particulars are needed to insure meaning and comprehension of generals; generalizations are needed to serve as suggesters and to make knowledge useful. There is considerable evidence that training in generalization is seriously neglected. The author believes that mathematics teaching can be improved if teachers will strive harder to train their students in proper habits of generalization.

The Association of Mathematics Teachers of New Jersey held their 32d regular meeting at the Atlantic City High School on Monday, November 12th, at 2 P.M., with Miss Josephine Emerson, of Kent Place School of Summit, N. J., presiding. The following program was given: "What Constitutes Relative Difficulty in Algebra," Andrew S. Hegeman, Central High School, Newark; "Mathematics and Religion," Harrison E. Webb, Market Street High School, Newark; "Unrealized Possibilities in Junior High School Mathematics," Miss Vera Sanford, Columbia University.

A STUDY OF PROGNOSIS IN HIGH SCHOOL ALGEBRA

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During the past two decades high school registration has increased many fold. The growth has been so rapid that in New York City, for example, the authorities have not been able to supply buildings fast enough to meet the new demands. Chief among the causes of this change is the Compulsory Education Law which compels many boys and girls to remain in school a year or two at least after their graduation from the elementary school. High school education has become the vogue and the high schools have therefore been forced to accept a large number of pupils who are not fitted for the various courses which are offered. The extent to which this condition holds is indicated by the number of failures each term. Commercial and vocational courses of various kinds have been introduced to take care of pupils whose needs are not met by the traditional subjects. The syllabi of the traditional subjects have been modified and simplified to meet the varying abilities and needs of the pupils. The effect of this tendency is seen in such courses as general science, general language and general mathematics.

Nevertheless, a large number of boys and girls who ought not to do so still enter the academic courses. They struggle with the academic subjects, failing and repeating, and then either change their course or, as most of them do, become discouraged and leave school after having wasted a year or two on work to which they should never have been subjected. If it had been possible to foretell fairly accurately the probable success or failure of these pupils in the various subjects, they would have saved valuable energy and time, they would have avoided the experience of unnecessary failure and they would have accomplished something worthwhile in work for which they are better fitted by nature.

Among the academic subjects referred to, algebra has been one of the leading stumbling-blocks to these misplaced boys and girls. For years it has been almost a tradition in certain high schools that from thirty to forty percent of the first year pupils should fail in elementary algebra; and this in spite of the fact that algebra followed immediately after the completion of eight years of arithmetic. About fifteen years ago one of the New York City superintendents suggested that algebra be begun after the first semester, so that the classes would not contain the poor material which drops out of school by the end of the first six months. The schools which adopted this change still found the traditional percentage of failure in the algebra classes. Some schools have in the recent past attempted some form of segregation in algebra, but the classification is based on teachers' estimates of work done over a period of weeks or months and is not as scientific as it should be. And even with this grouping many pupils are still wasting their time and no care is taken of those who should not have been permitted to undertake the work. What the schools need is an instrument that will test a pupil's ability in algebra and prognosticate his probable success or failure.

Such an instrument should take into account those factors that influence success in algebra achievement—ability to handle algebra situations such as the pupil meets in the algebra class, mastery of such prerequisites as arithmetic and reading with comprehension, school habits, environmental conditions, general intelligence, and the like. This study deals only with two of the factors mentioned—ability to handle algebraic situations, and school habits.

What factors are involved in a pupil's ability to handle algebraic situations? The pupils are told in the algebra classes that they are continuing in a generalized form the use of the operations which they learned in arithmetic, through the introduction of letters. Therefore, through the study of algebra we expect the student to develop the ability to (*a*) Understand the use of letters to represent numbers; (*b*) Substitute numbers for letters; (*c*) Evaluate algebraic expressions by substituting numbers for letters; (*d*) Apply the arithmetical operations to algebraic expressions; (*e*) Understand and to handle signed numbers; (*f*) Translate into algebraic symbols simple English

phrases and statements; (*g*) Solve problems. If the pupil has these abilities, the rest depends upon the learning process, which in turn is tied up with the method of presentation by the teacher.

In order that these abilities may be discovered, is it not the simplest thing to submit to the pupil some material in very elementary form which utilizes the learning process. For example, early in the year the algebra teacher finds it necessary to teach his pupils the meaning of various English expressions which they will need in the solution of problems, such as 'one number exceeds another,' 'the excess of one number over another,' 'the difference between two numbers,' 'one number is greater than another by a certain amount.' He explains and illustrates the meaning of these terms and continues with a short, effective drill. He also refers the class to the textbook and assigns the lesson to be done during the study period (in school or at home). In the assignment the pupil goes through the same process again of studying an explanation and illustrations thereof and then working out some exercises based upon them. A prognosis test composed of brief lessons and exercises following this same plan is, therefore, the instrument that calls for work most similar to that which the pupil will be expected to do throughout the year.

In addition to the purely algebraic abilities one must also consider whether or not the pupil is able to handle the necessary arithmetic which is presupposed. There is the ever-present cry among teachers of algebra that their pupils are not well-trained in arithmetic. How can they be expected to continue the arithmetical operations in generalized form, when they do not know the operations themselves? The reasons for the poor training need not be listed. The fact remains that a large number of pupils cannot handle fractions either common or decimal, and that lack makes their algebra work very difficult. There is evidence also of weakness in the very fundamentals of arithmetic. In a simple timed test given to 646 pupils in three New York City high schools, as part of a prognosis test, during the first two periods of the algebra year the following was the response:

Question	Percent of Pupils Correct	Percent of Pupils Wrong	Percent of Pupils Who Did Not Answer	Percent of Pupils Right of Those Who Answered the Example
(1) Multiply 254 × 69	72.5	27.0	0.5	70.0
(2) Divide 43/8987	83.0	16.5	0.5	83.0
(3) Divide .007/4.55	67.5	28.5	4.0	71.0
(4) Reduce 32/72 to lowest terms	83.0	13.5	3.5	87.0
(5) Add 3 2/5 and 1 7/8	62.0	31.0	7.0	67.0
(6) Subtract 5 5/6 from 6 5/8	45.0	42.0	13.0	51.0
(7) 3/49 × 2/5 × 8 3/4	35.0	35.0	30.0	50.0
(8) Divide 9 3/4 by 1 5/8	30.5	28.5	41.0	51.0
(9) 3/4 equals ?/36	23.5	27.0	49.5	48.0
(10) Add 3 yd. 2 ft. 4 in.; 1 yd. 5 in.; 1 ft. 7 in.	19.5	30.5	50.0	39.0
(11) From 7 gal. 3 qt. 1 pt. take 3 gal. 1 qt.	29.0	13.5	57.5	68.0
(12) What number is contained in both 15 and 12?	29.5	12.5	58.0	70.0
(13) What is the greatest number into which both 16 and 12 can be divided?	21.0	29.0	50.0	42.0

Five minutes was considered ample time for the thirteen questions. Disregarding the 0.5 percent that did not attempt the first two exercises, is it not significant that only 72.5 percent were able to do the multiplication correctly and only 83 percent the division? Is it not significant also that only 71 percent of the 96 percent who attempted the third one were able to do the division involving decimals? What an indictment of the elementary school instruction does one see in the results for numbers 5, 6, 7, 8 and 9—namely, the percentage of pupils who did not attempt them and the fact that half of those who did work them were wrong!

What is to be done in the algebra classes with pupils who prove to have a poor foundation in arithmetic? If this is discovered at the outset, provision can be made in due time in the form of special instruction during assigned periods or after school hours. If a prognosis test, therefore, contains a test in arithmetic, it will provide the necessary information, and time

may be taken to remedy the deficiences of pupils who are found to lack the necessary arithmetic ability. Those pupils who also lack the necessary algebraic ability may be eliminated from the study of algebra and assigned to some other work. Or, if the program exigencies do not permit that, then in the event of failure—which is bound to come—at the end of the term, the teacher can quite confidently advise the pupil to discontinue the study of algebra. In the case where the parent insists that the pupil remain in the course because of plans that he has made for his child, the teacher is in a position to confront him with evidence that cannot be disputed.

An attempt has been made to devise an instrument of the sort described above in the Orleans Algebra Prognosis Test, which consists of an arithmetic test followed by twelve parts covering the following topics: (1) Use of letters to represent numbers and substitution of numbers for letters in monomials. (2) Meaning of exponent. (3) Use of exponent in substitution. (4) Substitution in monomials involving exponents. (5) Substitution in binomials involving exponents. (6) Definition of like terms. (7) Algebraic representation in short simple questions. (8) Algebraic translation of simple statements. (9) Signed numbers. (10) Simple problems. (11) Algebraic addition of monomials involving coefficients and exponents. (12) Summary test. Each part (except the twelfth) consists of a brief lesson explaining and illustrating one unit of work followed by a test based on the lesson, both lesson and test being carefully timed. The authors feel that they have for the first time put into the hands of teachers of algebra a powerful instrument which will prove of benefit both to them and to their pupils.

The test is intended to measure the ability of the pupil to do the sort of work that he will meet in the classroom. This ability is undoubtedly one of the most important factors in success in studying algebra. Another important factor is the pupils' school habits such as attentiveness in the classroom, regularity of home-work, perseverance, industry, and the like. It will be readily admitted that these two factors are of great significance. It is of greater value to see how significant they are. In order to determine this the Orleans Algebra Prognosis Test was given to 250 pupils just before they began the study of algebra.[1] At

[1] This was a revision of the test which had been tried out the year before with approximately 300 pupils.

the end of the semester these pupils took an objective algebra test covering the work of the half year. In order to determine the extent to which they had formed certain school habits the teachers they had had the previous year were asked to fill out a copy of the New York Rating Scale for School Habits [2] for each pupil. The scores obtained by the pupils on the objective achievement test in algebra were taken as the criterion against which to validate both the prognosis test and the scores on the rating scale. In other words, the scores on the achievement test were taken as the measures of the achievement of the pupils in the half year's work in algebra. To the extent to which the achievement test was inadequate for this purpose, the prognosis test and rating scale will not be valid. As it turned out, the average score on the achievement test was rather low being only about 30 percent of the total possible score on the test. In other words the achievement test was too hard. This should be kept in mind in interpreting the following data.

The value of the prognosis test and rating scale can be determined by measuring the correspondence between the scores of those two and the scores on the achievement test. If the correspondence is close then these two devices have prognostic value. If the correspondence is not close then they are of little value for this purpose. A concise method of measuring the correspondence is the coefficient of correlation which may be obtained by means of a mathematical formula.[3]

For 120 of the above mentioned pupils the correlation coefficient between the Orleans Algebra Prognosis Test and the New York Rating Scale for School Habits on the one hand with the scores on the objective achievement test on the other hand, was 0.82. For another group of 130 pupils records were available on the

[2] Published by the World Book Company, Yonkers-on-the-Hudson, New York.

[3] There is no need here to go into a discussion of correlation. The reader who wishes to pursue this subject further is referred to Otis, A. S., *Statistical Method in Educational Measurement.* World Book Company, Yonkers-on-Hudson, New York. Perfect correspondence between two sets of scores results in a correlation coefficient of 1.00 Complete lack of correspondence results in a correlation coefficient of 0.00. A fair correspondence would result in a correlation coefficient of 0.80 or higher. For accurate prognosis the correlation should be above 0.95 but a correlation coefficient of 0.80 will give a very good basis for classifying and a fairly good basis for eliminating the very poorest pupils.

prognosis test and achievement test but not on the rating scale. For these pupils the correlation between the scores on the prognosis test and the scores on the achievement test was 0.71.

There are a number of factors that influence correlations such as the above but which can be controlled, only in part, if at all. Such factors are the effect, on the pupil's work, of absence, poor physical condition, incompatibility with the teacher, number of hours spent outside of school on remunerative work, home facilities for study, and so on. It is hardly possible that a teacher, even an excellent one, will as a result of a half year's effort get each pupil to do as much work as he should. The differential between the pupil's achievement under normal school conditions and what he could achieve under unusually good conditions would not be the same for all pupils and this variation might seriously affect the correlations given above. An adequate criterion will also reduce the correlation by not differentiating adequately between the achievement of the pupils. All these factors, as well as others, have the effect of lowering the correlations computed. It may therefore be safely said that the correspondence between the prognosis test scores and an adequate measure of achievement in algebra would certain be high enough so that the test may be used safely for both classification and the elimination of the poorest pupils. The use of a rating scale together with the prognosis test would enhance its prognostic value.

One might expect that a standard should be set for a prognosis test so that when a school uses the test it would know that pupils falling below that standard would have difficulty in passing the work in the subject, and pupils obtaining scores above the standard would be able to pass. This however, is not reasonable. A pupil's ability to succeed in school work depends on much more than his own ability and habits. The conditions of the school have their influence. Such are the preparation, ability and technique of the teacher, the relationship between teacher and pupils, the extent to which the particular subject is stressed in the school, and so on. As the result of the variation in these characteristics from one school to another it is not possible to set one standard on a prognosis test for all schools. In one school a score of 50 points on the prognosis test may be high, in another the same score may be low. Therefore, in using this

test, each school will have to determine its own standard. The best basis for this is the percentage of pupils that customarily fail in the subject. A detailed discussion of the interpretations and uses of the test results are given in the Manual which accompanies the test.[4]

It is worth while to compare the median and distribution of scores on the prognosis test for any school with the median and distribution of scores on the test for a large unselected group of beginners in algebra. Such comparison will enable the teacher, principal, or superintendent to determine how good or poor a group the algebra beginners form, what may be expected of them, whether the local standards are too high or too low. It will also provide the information needed as a basis for administrative and supervisory plans. Data for this comparison will be published in the manual for the test when a sufficient number of records are obtained.

ON TO CLEVELAND! THE FOURTH YEARBOOK OF THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS ON "SIGNIFICANT CHANGES AND TRENDS IN THE TEACHING OF MATHEMATICS THROUGHOUT THE WORLD SINCE 1910" WILL BE DISCUSSED! LOOK FOR THE TENTATIVE PROGRAM OF THE MEETING IN THIS ISSUE.

[4] The Orleans Algebra Test is published by the World Book Company, Yonkers-on-Hudson, New York. The published test contains several parts in addition to those listed in this article and each part of the test has been made longer. The correlation between the test scores and an adequate measure of achievement should therefore be appreciably higher than the correlations noted above.

THE FUNCTIONS OF INTUITIVE AND DEMONSTRATIVE GEOMETRY

By LAURA BLANK

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Intuitive geometry has been extensively emphasized and discussed in the last ten years usually in connection with general mathematics and the junior high school movement and but rarely in its relation to demonstrative geometry. It is useful and important, no so much so in itself, but rather in its connection with demonstrative geometry. The pedagogical books and journals have stressed intuitive geometry emphatically whereas they have briefly discussed demonstrative geometry, due, no doubt, to the fact that the latter subject has held its own for twenty-three hundred years among the subjects of pursuit in a liberal education. What is meant by intuitive geometry? Have the two subjects much in common? What is the field of each? What is the aim, purpose, or objective in the pursuit of each? Where in the school curriculum is the place of each? Are they subjects that can be fused, then taught simultaneously or should they be taught successively? Can one replace the other? It is not my intention to answer each of these questions and put an emphasis of finality upon whatever answers are suggested. Yet discussion of such questions tends to clarify opinions and theories on the part of instructors; it serves as a means of leading us to set up our own objectives and hence to work out our own plans as to subject matter, theories, and methods consistent with our own objectives. If each of us were to sit down alone to analyze introspectively and carefully our teaching methods and results, we should possibly find that at times we flounder in the two aspects of geometrical instruction to no purpose. Perhaps some of the text-books, written as they are, not from purely altruistic motives, aim to meet at one and the same time the separate and distinct demands of the two subjects. Hence such texts further confuse our opinions and positions relative to the two sorts of geometry.

Intuition means immediate apprehension or cognition, the faculty or power of such apprehension; knowledge obtained or the power of knowing without recourse to inference or reasoning; innate or instinctive knowledge; a quick or ready insight. Intuitive gemetry means therefore that body of geometrical concepts, facts, truths which one is capable of discovering and using without complex deduction or reasoning. A demonstration, on the other hand, is a course of reasoning showing that a certain result is a necessary consequence of assumed premises—these premises being definitions, axioms, postulates and previously established propositions.

Teachers and students of mathematics, because of the recent and current emphasis upon intuitive geometry regard it as distinctly modern, as an invention or discovery of the last two decades. However, on the contrary, the Greeks, before Euclid, have been found in many instances to have stated a theorem, constructed the appropriate figure and added merely the laconic and startling comment, "Behold!" In some cases mathematicians of later times have spent a lifetime upon such a theorem before a satisfactory demonstration was completed. Thales, for instance, discovered and stated without scientifically proving the equality of vertical angles. Plutarch attributes to Thales the feat of measuring the height of the pyramids by means of the observation of the length of shadows, several hundreds of years before the establishment of the general theory of similar triangles and proportions. Before the time of Euclid, in the beginnings of geometry, many qualities and characteristics of geometrical figures were doubtless assumed as obvious or verified by measurement, as for example, the equality of the opposite sides of a parallelogram. The history of geometry convinces us that the association of the lengths, 3, 4, 5, with the sides of the right triangle was discovered and used hundreds of years prior to the demonstration of the theorem now called Pythagorean. History therefore bears out the fact that orderly scientific demonstrative geometry is an outgrowth of intuition, or immediate apprehension or quick and ready insight in early and fundamental geometric concepts.

Moreover, as far as teaching is concerned, mensuration which is akin to intuitive geometry, a kind of intuitive geometry, in fact, in which the emphasis is laid almost entirely upon applica-

tion, has been included in the mathematics of the eighth grade for a score of years. Text-books of twenty years ago showed a cubic yard as composed of 3 layers, of 9 each, of small cubes, each small cube intended to represent one cubic foot. One method of working out the formula for the lateral area of a cylinder of revolution was illustrated by showing a rectangle as it was being rolled so that one edge was brought around so as to just coincide with the corresponding edge.

However the intuitive geometry developed recently is usually more conducive to reasoning. The intention is to endeavor to convince the pupil of the truth of theorems at all times, not merely by application, but by insight concerning some sort of illustration, experiment or construction; moreover to suggest inferences and even brief informal proofs involving perhaps two or three steps of reasoning. It is the theory of all proponents of intuitive geometry that none of the work shall be done on the authority of another, but that all theory, formulas, and other authorities shall be discovered and tested out by means of some sort of individual or class investigation however simple and elementary in character. Among the topics of study in intuitive geometry are: Direct measurement of distances and angles by means of a linear scale and protractor; the approximate character of measurement; an understanding of estimates of results of computations, of "rounding off" of arithmetical calculations and an understanding of "significant figures"; the areas of simple rectilinear figures and the circumference and area of a circle and the surfaces and volumes of the simpler solids; indirect measurement by means of drawings to scale; use of paper cutting and squared paper; simple geometric constructions with ruler, protractor and compasses; familiarity with the equilateral, isosceles, and right triangle and the theorem of Pythagoras and the theorem concerning the sum of the angles of a triangle; and the informal study of the idea of similarity. Moreover a course in intuitive geometry often includes the study of the trigonometry of the right triangle with the use of the tangent function and possibly the sine function.

The subject matter of both intuitive and demonstrative geometry *may* comprise therefore all Euclidean plane and solid geometry. The marked difference in the two courses lies in the treatment of this subject matter. However, because of its very

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character, intuitive geometry is more restricted in subject matter than demonstrative geometry for many theorems of Euclid are regarded as too complex to permit of intuitive treatment.

Even demonstrative geometry has undergone change in the last two decades. No longer is the attempt made to teach all of the first five books of Euclid in plane geometry, but rather syllabi have been worked out following the plan and sequence of Euclid, yet limiting the number of theorems for demonstration. Certain of the more obvious theorems are explicitly postulated thus extending the list of assumptions, yet in no way destroying the sequence of reasoning so essential for genuine demonstrative geometry. From a strictly logical aspect it is not necessary to restrict the list of assumptions such as axioms and postulates to a minimum. Applications of graded difficulty from varied fields and interests are emphasized and made quite a large part of any course in demonstrative geometry to-day.

Intuitive geometry is distinctly a junior high school subject. It is an exploratory and to some small extent a diagnostic subject. It provides a preview or survey of a new, broad, and interesting field of learning. It may be made, in the hands of a sympathetic enthusiastic and well trained instructor an inspirational course. However, by no means all of the classes in intuitive geometry, which one may chance to observe to-day, are doing inspirational work under inspired teachers. One sees much "busy work," and many projects fit merely for primary instruction. Only if continual emphasis is laid on geometric form in the plane and in three dimensions with respect to shape, size and position, if much opportunity is provided for exercising space perception and imagination, if the work is planned so as to extend relations in the plane to three-dimensional space, to bring out geometrical relations and simple logical connections, to suggest inferences and to draw valid conclusions from relations discovered, to provide a gradual informal approach and a foundation for space concepts and relations leading to the subsequent work in demonstrative geometry—if a course in intuitive geometry is of this type and is taught by an enthusiastic teacher, capable of giving an equally good course in demonstrative geometry, then it is in truth a course worthy of a place in the junior high school curriculum. Indeed such a course may be made of great value in itself to those pupils who are incapable of profit-

ing by a course in demonstrative geometry. It provides a variety of experience and information which the pupil fails to get elsewhere, in the elementary or junior high school, which is essential for his later contacts, development, and adjustments to the world of space around about him.

Demonstrative geometry is on the other hand uniquely a senior high school subject, though it may well be preceded by a reasonable amount of informal geometry of an experimental, intuitive and constructive nature. Geometric facts previously inferred intuitively may be used as the basis for demonstrative work, *only* provided such practice can be brought about so as not to lead to loose, careless, inexact work and habits. In observing classes where it is the custom to base much of the demonstrative work upon geometric facts previously discovered by intuition, one finds, as a rule, that the theorems are inaccurately quoted or merely suggested, the reasoning is loose and careless, in fact, one feels a constant lack of all of those characteristics for which formal demonstrative geometry is valued. The practice, infrequently followed, of introducing formal geometrical demonstration prior to the tenth grade is conducive of the same sort of results. Pupils are either too immature to understand demonstration in its formality, or if there is some understanding, the practice in formal demonstration is so little pursued and so lightly stressed as to bring about confusion, carelessness and laxness, so wholly foreign to all that formal demonstration connotes. The custom of basing the demonstrative work upon facts previously inferred intuitively is not intended to preclude the possibility of giving at a later time rigorous proofs of these same facts previously inferred, the object here being explicitly stated as not discovery, but demonstration. The principai purposes of instruction in demonstrative geometry are: To exercise further the spacial imagination, to familiarize the pupil with the great fundamental theorems and their applications, to develop the appreciation and understanding of a demonstration, deductive, indirect or analytic, and the ability to use this method of reasoning when it is applicable, and to form habits of exact, precise and succinct statement, of the logical organization of ideas and of logical memory. Moreover the pupil should be led to see the vital relation of the subject to the development of civilization.

Smith and Reeve, in their new book, "The Teaching of Junior High School Mathematics," assume a rather conservative position concerning the subject, saying: "The real purpose of the subject [demonstrative geometry] is suggested in the word 'demonstrative' rather than by 'geometry.' The mere utilities of geometry have already been acquired before the pupil begins, if he ever does, the work in what is to him an entirely new field —that of logical proof. Nowhere in his previous training, nowhere else in his elementary education, does he come in close contact with a logical proof. The chief purpose of this part of mathematics, then, is to lead the pupil to understand what it is to demonstrate something, to prove a statement, logically, to 'stand on the vantage ground of truth.' He sees a sequence of theorems built up into a logical system and he sees how this system is constructed the result being a basis of proved statements which he can use for establishing further proofs as a lawyer proceeds to construct his case or a speaker to construct an argument."

Carl Sandburg, in his "Abraham Lincoln," says of the lawyer, Lincoln, "He heard the word 'demonstrate' and said to himself: 'What do I do when I demonstrate, more than when I reason or prove? How does demonstration differ from other proof?' He looked in Noah Webster's dictionary and learned that demonstration is 'proof beyond the possibility of doubt.'

"The definition didn't satisfy him; he went to all the dictionaries and books of reference he could find for the meaning of the word 'demonstrate' and in the end said to himself that their definitions meant about as much to him as the color blue when explained to a blind man. He said to himself, 'Lincoln, you can never make a lawyer of yourself until you understand what "demonstrate" means.'

"He bought 'The Elements of Euclid,' a book twenty-three centuries old. It began with definitions, such as (1) A point is that which has no parts, and which has no magnitude; . . . ."

"Quietly, by himself, he worked with these definitions and axioms. The book, 'The Elements of Euclid,' went into his carpetbag as he went out on the circuit. At night, when with other lawyers, two in a bed, eight and ten in a hotel room, he read Euclid by the light of a candle after others had dropped off to sleep.

"Herndon and Lincoln had the same bed one night, and Herndon noticed his partner's legs pushing their feet out beyond the footboard of the bed, as he held Euclid close to the candlelight and learned to demonstrate such propositions as: 'In equal circles, equal angles stand on equal arcs, whether they be at the centres or circumferences.' . . ."

"He was trying to organize his mind and life so that he could not accuse himself, as he had accused President Polk, of being 'a bewildered, confounded, and miserably perplexed man.' He wanted to be simple as the alphabet, definite as the numbers used in arithmetic, sure as the axioms or common notions that are starting-points of Euclid. Had he trusted too much to his feelings, and not reasoned, proved, and demonstrated his propositions clearly enough in his own mind before speaking them during his term in Congress? He wasn't sure."

Sandburg continues with reference to facsimile of some of Lincoln's notes written at this time: "Lincoln writes loose notes trying to reason in politics and human relationships with some of the absolute quality of mathematics. To *prove* a thing isn't enough; he wants to *demonstrate*. By such tests and rehearsals he aims to be trained so that he can meet all comers in debate and overthrow them. He is dropping away from the horseplay and comic sarcasm of his oratorical style of earlier years."

Finally with reference to Euclid, it is and doubtless will remain the greatest mathematical textbook of all time. Scarcely any other book except the Bible can have circulated more widely the world over or can have been more studied and edited. Whether the young adolescent student of this great work finds himself or herself in the rôle of an explorer, a discoverer, resorting to instinct, intuition or apprehension, emulating thus the early experience of the human race, with reference to geometry; or he, as an older boy in his middle teens, studies it finding it to be a great monument—though not altogether perfect—to the human intellect, an almost perfect science, from which he may gain habits of exact reasoning and mental poise, surely at some time in his mental development he will agree that "the general experiences of mankind upon the subject are sufficient to justify us in demanding for it a reasonable amount of time in the framing of a curriculum."

ABILITY CLASSIFICATION IN NINTH GRADE ALGEBRA [1]

By L. E. MENSENKAMP

High School, Freeport, Illinois

Certain tendencies which have long been operating in our schools have by their cumulative effect brought about a change in the ninth grade school population which has made classification into ability groups and other devices for adjusting instruction to the varying needs of the individual pupil more important than ever before.

Let us see briefly what this change has been. Thorndike, in his *Psychology of Algebra* (p. 4), states that "For every one hundred children who reached fourteen there were about three and one half times as many beginning high school in 1918 as in 1890." Byrne [2] estimated that in 1918 about 40 percent of that part of our population which was of ninth grade age was actually enrolled in the ninth grade. The tendency each succeeding year for a larger proportion of those completing the eighth grade to continue on into high school has operated so effectively that in certain communities at the present time only a small proportion of the pupils drop out of school between the eighth and ninth grades. Thorndike says further in the place cited above, "We know that education is selective and that the correlation between native capacity and continuance in school to higher and higher grades is positive." The facts concerning the intelligence ratings of those who drop out of school at the various educational levels support this statement.

There can be only one conclusion from the above considerations, namely, that our educational system is becoming steadily less selective with regard to mental ability. As a result the high school teacher of to-day is confronted with a greater range of talent in his classes than ever before, and the extension has taken place at the lower end of the ability scale by the addition

[1] Presented at the Third Annual Conference of Teachers of Mathematics at the University of Iowa, Iowa City, October 13, 1928.

[2] Quoted by E. L. Thorndike, *Psychology of Algebra*, p. 4.

of a type of pupil who formerly was not inclined to seek a high school education.

The change we have just described has been accompanied by a steady increase in our total population, and the combined result has been a tremendous increase in the enrollment in our secondary schools. This has led to crowding in many communities, and teachers have been called upon to handle many more pupils per class than heretofore. Under these conditions one of the most effective ways to provide for individual differences is by the classification of pupils into ability groups so as to permit the adaptation of teaching methods and the course of study to the particular needs of each group.

It is proposed in this paper (1) to describe and evaluate the methods of classification which have been employed in the writer's own school, and (2) to discuss methods of adjusting instruction to the ability of each group.

The scheme of classification which we are going to describe is one which should prove satisfactory in schools where there are a hundred or more ninth grade pupils enrolled. With suitable modifications it can be used for a considerably smaller number of pupils than this. It contemplates dividing the pupils into four groups, three of which are assigned to algebra and the fourth to a class in ninth grade arithmetic. The number of pupils assigned to each of the algebra groups should be somewhat as follows:

(*a*) Superior group 30–40
(*b*) Medium group 25–30
(*c*) Inferior group 15–25

It goes without saying that no mention is ever made to the pupils of "superior" and "inferior" groups. A letter designation is given to each of the various classes during the enrollment days to insure pupils being assigned to the proper group, but no ability characterization of a group is ever made to the pupils.

In addition to the three above groups we have a fourth one which may for the purposes of this paper be characterized as a "very inferior" group. This group, comprising about fifteen or twenty percent of all entering freshmen, consists of those who, on the basis of eighth grade records and general intelligence rat-

ings, give very little promise of success in algebra even under the most favorable conditions. These students are advised to substitute a course in ninth grade arithmetic for algebra with the understanding that if they make a satisfactory record in arithmetic, they may take algebra the following year if they so desire. It is felt that a good general course in arithmetic is well suited to the limited abilities and probable later-life requirements of such students, and at the same time the additional training it provides should greatly increase their chances of being able to cope successfully with the traditional high school mathematics courses.

In order to provide for flexibility of classification among the algebra sections it is desirable to arrange them, so far as possible, in groups of three at the same hour so that some necessary shifting and adjustment may be done later without upsetting the rest of the student's schedule. We have found that not a great many such adjustments are necessary, however.

Such a scheme of classification as that outlined above presupposes that there are means available for predicting with at least a fair degree of accuracy the caliber of a student's future performance in ninth grade algebra. The predictors which have been used in this study are (1) the second semester grades in eighth grade arithmetic and (2) the Intelligence Quotients (I.Q.'s) given by the Otis Self-Administering Tests of Mental Ability.

A critical evaluation of the effectiveness of these predictors is desirable. The means for making this evaluation is available in the well known methods of correlation in mathematical statistics.

The method of simple correlation is so well understood and there are so many expositions of it now available [3] that there is no need to discuss it in detail here. The product-moment coefficient of correlation, r, calculable from a suitable formula, is a satisfactory measure of the degree of statistical dependence of one variable upon another provided the regression is *linear*. In the case of *non-linear* regression, that is, in case the pairs of

[3] For a clear exposition of the underlying theory of the correlation methods used in this paper see H. L. Rietz, *Mathematical Statistics* (the third Carus Monograph), pp. 77–101. Detailed methods of calculation may be found in the texts on educational statistics, *e.g.*, K. J. Holzinger, *Statistical Methods for Students in Education.*

variates, when represented as points on a scatter diagram, tend to concentrate about some more or less irregular curve, the *correlation ratio* must be used as a measure of the degree of statistical relationship. For each correlation table there are two correlation ratios, one measuring the dependence of the first variable upon the second, and the other measuring the dependence of the second on the first. To decide whether the correlation coefficient or the correlation ratio is the appropriate measure to be used in any given case we apply Blakeman's Test for Linearity of Regression.[4] The significance of the correlation coefficients and other frequency constants can only be properly gauged when they are accompanied by the values of their probable errors of random sampling.

When two predictors are to be used a regression equation involving three variables is needed. Let X_1 represent a student's first semester algebra mark, X_2 his second semester eighth grade arithmetic mark, and X_3 his Intelligence Quotient as given on the Otis test. We then determine by the methods of mathematical statistics the values of the constants *a*, *b*, and *c* which will give the best possible prediction of an algebra mark from the equation

$$X_1 = aX_2 + bX_3 + c.$$

The predictive value of this equation may be measured by the size of the multiple correlation coefficient. This coefficient, which is denoted by the symbol, $r_{1.23}$,[5] tells us to what extent the algebra marks as estimated from the arithmetic marks and the I.Q.'s by means of the above equation correlate with those actually received by the students at the end of the first semester.

The methods just described were used to study the records of 406 students who have taken beginning algebra during the past three years. The results are given in the table below. In this table we give as the First Correlation Ratio the one which measures the dependence of the first mentioned variate upon the second.

[4] J. Blakeman, "On Tests for the Linearity of Regression in Frequency Distributions," *Biometrika*, Vol. IV (1905), pp. 333–350.

[5] Frequently written $R_{1(23)}$.—The Editor.

Correlation Results. $N = 406$

Variates Correlated	Correlation Coefficient	First Correlation Ratio	Second Correlation Ratio
Algebra—Arith.	.54 ± .024	.56	.56
Algebra—I.Q.	.42 ± .027	.44	.43
Arith.—I.Q.	.35 ± .029	.38	.37

Application of the short form [6] of Blakeman's Test to the data in our table shows that all of the regressions are substantially linear, and the use of r as a measure of correlation is justified.

There is reason for believing that the true correlations between the abilities we have been investigating are somewhat higher than those we have found.[7] Two factors are largely responsible for lowering our correlations. The first is the unreliability of teachers' marks. The effect of this factor is likely to be especially pronounced in schools where ability sectioning is practiced. The invariable tendency of teachers to mark good pupils too low and poor pupils too high in classified sections is a matter of common observation. It should be possible, however, for the mathematics teachers in a given school to do much toward raising the reliability of their marks by concerted efforts to secure a more uniform and objective marking system. The second factor tending to lower our correlations arises from the fact that we have excluded many of our less gifted pupils from the algebra course. Selection of this kind curtails the range of the distribution and therefore causes a decrease in the size of the correlation coefficients.

We next inquire as to what gain in predictive value may be attained by using the best linear combination of our arthmetic marks and I.Q.'s as explained under the discussion of the multiple correlation coefficient. It turns out that our regression equation is

$$X_1 = 0.56X_2 + 0.19X_3 + 14.3. \qquad (1)$$

For use in this equation letter marks in arithmetic have been converted into numerical grades comparable to those used in the

[6] H. L. Rietz, *Simple Correlation* (Handbook of Mathematical Statistics), p. 131.

[7] P. M. Symmonds, *Special Disability in Algebra* (Columbia University Contributions to Education No. 132), Chapters V, VI, and VII.

high school. The scale is on the basis of 100, and 75 is the passing grade. One must be careful not to interpret the coefficients of X_2 and X_3 in equation (1) as measuring the relative values of the arithmetic marks and the I.Q.'s as predictors of the algebra mark. It must be remembered that the two variables are expressed in *different* units. This disparity in units is responsible for part of the difference in the size of these coefficients.

The multiple correlation coefficient which measures the amount of agreement between the algebra mark actually earned by the student and that predicted by equation (1) turns out to be $r_{1\cdot 23} = .60$.

Another way in which to judge the predicative value of our regression equation is by calculating the dispersion of the estimated values. The probable error of an algebra grade predicted by equation (1) is found from the formula

$$\text{P.E.} = .6745\sigma(1 - r_{1\cdot 23}^2)^{\frac{1}{2}}, \tag{2}$$

where sigma is the standard deviation of all the algebra grades. It results that P.E. $= \pm 4.1$ percent. If we assume that the algebra grades which correspond to assigned values of the arithmetic grade and the I.Q.'s are distributed normally, we may make the following statements to illustrate the meaning of our result. If the grade predicted by equation (1) is 85, for example, the chances are even that the grade received by the student will fall between 89.1 and 80.9. By reference to a table of values of the probability integral we find that there is only about on chance in ten that a student whose predicted grade is 75 (the passing grade) should actually receive a grade of more than 85. There is less than one chance in fifty for this same student to receive a grade of over 90.

If we use only one predictor, a regression equation containing two variables will be needed. The probable error of estimate involved in using such an equation is found from a formula exactly like (2) except that the simple correlation coefficient appears in place of $r_{1\cdot 23}$. From this formula it is found that the probable error of estimate when the arithmetic grade alone is used as the predictor is ± 4.3 per cent. Such estimates are therefore only slightly less accurate than those based on arithmetic and I.Q. combined.

Whenever predictions are made by means of regression equations of either two or three variables, it is advisable to save time and avoid numerical computation by resorting to graphic methods. Thus, to evaluate equation (1) an alignment chart consisting of three parallel scales can easily be constructed. Then it is only necessary to place a straight edge on the given points on the arithmetic and I.Q. scales, respectively, and read the algebra grade from the point where it crosses the algebra scale. In the case of a two variable equation the graphic chart would simply consist of two scales, one drawn on each side of the same straight line in such a way that predicted values of the one variable fall directly opposite assigned values of the other.

Our results indicate that whenever a student's record in eighth grade arithmetic is such as to place him in the lower fifth of a large, representative group of students of this subject, and at the same time his I.Q. is below 90, the odds are so great against his achieving any real success in the algebra that it seems advisable for him to take instead a good course in ninth grade arithmetic. The latter type of course seems better suited to his needs in later life. In rare cases where there is reason to doubt the correctness of our estimate of the algebraic aptitude of an individual as arrived at in the way described, further facts regarding his ability should be sought.

Probably the best basis for making a classification in second semester algebra would be the first semester grade in this subject. For 85 pupils who passed the course in first semester algebra the first time they took it we found a correlation between the first and second semester grades given by $r = .66 \pm .045$. If this value is corrected for the selection [8] which results from the elimination of those who failed the first course from our data, the correlation for the unselected group would be given by $r_u = .74$.

The question arises as to possible alternative bases of classification in first semester algebra. One of the most promising of these seems to be the Stanford Achievement Test in Arithmetic. In a study entitled *An Analysis of Algebraic Abilities*,[9] J. P. McCoy reports a correlation between scores on this test and alge-

[8] For a discussion of the effect of selection together with the necessary correction formulas see T. L. Kelley, *Statistical Method*, pp. 223–230.

[9] In an unpublished Ph.D. thesis, University of Iowa, 1924. Results quoted in Ruch and Stoddard, *Tests and Measurements in High School Instruction*, p. 82.

braic ability as measured by the combined scores on the Hotz and Douglass Tests to the extent of $r = .59$ for a population of 175 students from three high schools.

Thorndike[10] has suggested a prognostic battery of tests for algebra consisting of (1) a missing term in arithmetical series test, (2) a sentence completion test, (3) a test in arithmetical problems, and (4) a test in disarranged numerical equations. The present writer is not aware of any experimentation which has been carried out with such a test.

We may now state certain general conclusions regarding our mechanism of classification as follows:

1. The values of the correlation coefficients and the probable errors of estimate indicate that our predictions are not sufficiently accurate for us to expect perfect homogeneity of talent within our groups. Some provisions for flexibility of classification should accordingly be made.

2. Judging from published correlations bearing on the prognosis of algebraic ability, however, our method of classification seems to be about as accurate as any other that is now available.

The procedure of advising the less gifted pupils to substitute a course in arithmetic for algebra should effect a decided reduction in the number of failures in ninth grade mathematics. This is borne out by the results in our own school where we find that we now have only about one third as many failures in algebra as we did under the old system when it was required of all students. A possible objection to our plan is that it permits certain students to finish high school who are not able to meet the college entrance requirements in mathematics. This objection is met by requiring any student in this group who intends to go to college to follow their arithmetic with the usual courses in algebra and geometry. As a matter of fact the difficulty is more apparent than real because the less gifted student usually gravitates toward the practical, vocational courses in high school and almost never goes to college.

We now turn to the second part of our discussion, namely, the adjustment of instruction to the ability of the group. We shall be mostly concerned here with the slow group because it represents the most difficult problem. If we are to eliminate failure and discouragement, we must limit this group to the tasks at which most of its members are able to succeed.

[10] E. L. Thorndike, *Psychology of Algebra*, p. 216.

The superior group, on the other hand, with the help of good teaching should be expected to do everything in the better grade of text-book without much trouble. Their year's course should meet fully the requirements set forth in the College Entrance Examination Board's Algebra Report. Group teaching and the explanations in the book should meet their needs to a greater extent than in the case of the other two groups. Gifted pupils should be a constant challenge to their instructor to develop in them all the mathematical insight and skill of which they are capable. They are clearly entitled to an educational opportunity in algebra commensurate with their superior ability.

In connection with the adjustment of subject matter to the slow group certain general observations may first be made. A book with differentiated courses of study outlined for the various groups is to be preferred. Even with such a book certain selections and eliminations will probably be necessary. The algebra teachers in a school should get together and outline a detailed course of study suited to the needs of this group. In making this outline the following principles should be kept in mind:

1. Emphasis should be on the fundamental and the useful.
2. Extremely complex and abstract material should be eliminated.
3. A close contact with arithmetic should be maintained.

Let us see how these principles may be applied to some important parts of the subject matter of algebra.

As many as fifty percent of the more difficult verbal problems in certain texts may have to be eliminated. In particular, it seems desirable to exclude many problems which are purely numerical or which deal with highly artificial situations. We give the following as illustrations:

1. What number added to the numerator of the fraction 7/8 will make the new fraction 5/9 of the numerator which was added?
2. Eight years ago 5 times A's age was 19 more than 3 times B's age. Four years hence 6 times B's age will be 28 less than 8 times A's age. Find the age of each now.
3. A is 13 years younger than B; but 7 years from now 6 times A's age will exceed 3 times B's age by 33 years. Find the age of each now.

It is rather doubtful if problems like these have any very great appeal to the average student, to say nothing of the non-intellectual type we are discussing.

Among the types of problems to be emphasized with such a group we may mention those pertaining to business and finance, those pertaining to mensurational geometry, and those dealing with other concrete situations and practical applications.

Work with the formula and the graph constitutes excellent material for a slow section. The formula represents a practical phase of algebra, and is closer to arithmetic than many other parts of the subject. Graph work, especially that dealing with statistical graphs, should prove interesting and worth while.

Algebraic technique should emphasize the fundamental processes. Complicated algebraic manipulations should be omitted. For example, the four fundamental operations with fractions should be restricted to simple cases. Standardized practice exercises will help to establish the fundamental skills.

So far we have been chiefly concerned with the subject matter of algebra; let us now very briefly consider the question of methods suited to a slow group.

The personality of the teacher is an important factor in the situation. He must be temperamentally qualified to handle such a group. This means that he must be enthusiastic and energetic, and at the same time, patient and sympathetic. He must strive to avoid causing his pupils to feel that they are inferior. The maintenance of a satisfactory morale is very essential. If, as a result of repeated failures and incessant criticism on the part of the teacher, the class develops a mental set antagonistic to everything connected with algebra, the battle is lost. The teacher must always bear in mind that because of limited natural endowments this group should not be expected to attain the same level of proficiency in algebra as the superior or even the medium group.

Progress must be at a rate sufficiently slow to make real success possible. Some time could profitably be given to supervised study and the teaching of effective study habits. It is important to make a prompt diagnosis and correction of individual difficulties. This is a big task in a slow section because here these difficulties are more numerous and varied and the pupil requires a greater amount of detailed attention to get them straightened out. Extensive use must therefore be made of individual instruction. If possible, the size of the class should be kept small enough so as to give adequate opportunity for this. Whenever

difficulties appear which show lack of mastery by the group as a whole, the topic should be carefully re-taught. Constant reviews are most important.

Finally, use should be made of the human interests, such as rivalry, the love of social approval, and the like. Competition with others and with the pupil's own best record should be encouraged.

NATIONAL COUNCIL PROGRAM AT CLEVELAND, FEBRUARY 22 AND 23, 1929

The final program of the National Council of Teachers of Mathematics will appear in the February issue of the Teacher. However, the members of the Council will be interested to know that Professor C. H. Judd of the University of Chicago will speak on "Informational Mathematics *vs.* Computational Mathematics," that Professor C. N. Moore of the University of Cincinnati will speak on "Mathematics and the Future," and that Professor Louis C. Karpinski of the University of Michigan will speak on "The Permanent Nature of the Necessity for Mathematics in the Secondary School."

Other speakers will be Professor W. W. Rankin of Duke University, Mary S. Sabin of Denver, C. H. Lake, Assistant Superintendent of Schools of Cleveland, Fred N. Burroughs, President of the Cleveland Mathematics Club, Dr. Vera Sanford of the Lincoln School of Teachers College, Columbia University, Agnes Rowlands of the Jamaica Training School, Mrs. Florence Brooks Miller of the Fairmount Junior High School of Cleveland, and W. D. Reeve, editor of the MATHEMATICS TEACHER and the Fourth Yearbook.

The business session of the Council will begin Friday morning at the Hotel Statler and will run through the day. The formal program will begin Friday evening and will last through Saturday, closing with the annual banquet on Saturday night.

GEOMETRIC AIDS FOR ELEMENTARY ALGEBRA[1]

By ALBERTA S. WANEMACHER

Hutchinson High School, Buffalo, N. Y.

When Doctor John R. Clark's small son in a beginning class in demonstrative geometry was asked by his father where he had learned a certain geometric principle, he replied with a tone of disgust.

"Why, in the seventh grade where I should have learned it."

Perhaps our pupils could not have answered in that way, but I know that we are all looking forward to the time when work in intuitive geometry will begin at an earlier date than at present. Arithmetic and geometry developed simultaneously, the former through the need of counting, the latter through the need of measuring, and on the basis of this historical fact, the makers of the more progressive syllabi are introducing first lessons in informal geometry into the very early grades.

Seventh grade intuitive geometry is not my topic, but to those of you who are interested in curriculum construction problems, I would recommend *A Tentative Syllabus in Junior High School Mathematics,* just published by the New York State Department. As I have just said, seventh grade intuitive geometry is not my topic, nor am I concerned this afternoon primarily with that geometry which, if learned before the beginning of the tenth year, would make the approach to demonstrative geometry more direct and welcome to all the teachers of that subject. Those expecting a discussion of fusion or correlation, three days of one and two of the other, or that a unit of demonstrative geometry be recommended for the second term of the ninth year, are to be disappointed.

Geometric aids for elementary algebra, or geometry as a tool to be used whenever, by its use, we feel that there is a gain in interest and understanding, is my topic. I am bringing to you suggestions which have been tried, with at least some degree of success by teachers in the department at Hutchinson. Our pur-

[1] Delivered at a recent meeting of the Buffalo Mathematics Teachers.

pose has been, not to invent a geometric situation for every algebraic topic, but rather to use geometry, and by geometry of course, I mean that which would come under the heading of intuitive geometry, to illustrate algebra, and to give certain algebraic topics a concrete background. We do not claim to have exhausted the possibilities, and hope during the second half of this hour to have many additions given from your personal experiences.

You may argue that certain parts of this work sound simple, but when I consider the Freshman boys of my study room who are either now studying algebra or have studied it, and whose ages range from ten years up, yes, ten, with the majority under fourteen, I wonder whether the simplicity of algebra teaching is to be condemned. You may say also that there is not time for this geometric work, and I would answer, that time spent here is time saved elsewhere.

Dr. T. P. Nunn of England, one of the leaders of progressive mathematics of that country, an educator of prominence and a man who is looked up to by many of the mathematics teachers in the United States, says that the first lessons in algebra should teach the use of the formula. That opinion, I believe, at the present time, is pretty generally accepted.

However, there are formulas and formulas. What interest does an eighth or ninth grade boy or girl take in a formula plucked full-grown from the pages of a text-book of Physics, or General Science, or Mechanics, or what not? Let me read to you a few sentences from the aforementioned syllabus on the topic, "The Formula."

"Since the formula represents one of the most elementary means of introducing functional thinking into the elementary curriculum, it has become a central topic of algebra. It may be questioned, however, whether it is sound to regard the formula as the unique and distinctive contribution of elementary algebra, especially if its pedagogic purpose is not clearly recognized. At any rate, there is danger that the manipulation of formulas, namely their evaluation and their transformation, may become a merely mechanical routine. If the constant stressing of the formula in the recent literature on elementary mathematics should have this result, the ultimate effect would be merely that of replacing one formalism by another." As was stated above,

the national committee very wisely cautioned teachers against the danger resulting from the customary manipulation of symbols. It insists, moreover, that the "ability to understand the language of algebra, and to use it intelligently, and the ability to analyze a problem, to formulate it mathematically, and to interpret the result, must be dominant aims." In other words, intelligent comprehension and interpretation are far more important than mere manipulation. This means that the process of generalization which leads to the discovery of the formula would seem to be more important than the evaluation or transformation of symbolic statements. Of course, not every formula introduced into elementary algebra should be actually derived. Nevertheless, extensive consideration of formulas not discovered by the pupil is likely to create the very attitude which the national committee has criticized so strongly. The simplest mensurational formulas of geometry seem to represent the best point of departure. Perimeter formulas, as well as area and volume formulas, can easily be made plausible and can be applied to everyday problem situations.

Let me explain briefly the use of three very simple instruments in the construction of simple formulas, namely: toothpicks, string, and squared paper. Given these three materials, the possibility of building up formulas is almost unlimited.

With the toothpicks, regular polygons of three, four, five, or more sides can be quickly and easily laid out on the desk. As soon as the term perimeter is understood, statements about the perimeters of these figures can be written, then shortened into the formulas $P = 3s$, $P = 4s$, $P = 5s$, and so on. Thus, algebra as a shorthand language is introduced. By substituting values for s, the corresponding values of P can be found, these values may be integral or fractional, common, or decimal. Likewise by taking values of P and finding the corresponding values of s, the first type of simple equation is solved. Generalizing follows with the resulting formula $P = ns$. Place two toothpicks on each side, that is, double the length of each side, and again build up these regular polygons. Then compare and note the relationship between the first set and the second set as to the shape and length of perimeters. This, of course, is very simple functional thinking, but forms the basis upon which to build. If functional thinking is to be the unifying principle of all mathematics, we

would certainly lose a real opportunity if it were not introduced at this point.

One of the teachers asked each pupil to bring to class a piece of string to use at the board as a compass. With this string, the pupils measured the dimensions of the covers of their books and other rectangular objects at hand, and then on a working line at the board laid off these lengths. They checked their work by laying the book along the line. Then by using the letters l and w on the corresponding line segments, the formula $p = l + w + l + w$ was simply derived. Will pupils with this training be as likely later to make the mistakes of using $p = l + w$, or of writing $p = l \times w$, or $p = 2l + w$, or any other of the common mistakes made when perimeter is required?

What better place than right here to have a discussion of simple measuring instruments? It is true that it takes special training and attention to teach pupils to measure to the nearest millimeter or angles to the nearest degree, and to understand and appreciate the approximate character of all measurements; but the results of this time spent are surprisingly satisfactory.

This work leads naturally to the addition and subtraction of line segments. Such statements as $4a + 5b$, or $4c - 2c + f$, where those letters represent given line segments can be shown. The pupil then sees that $3a$ means a taken three times and added. With this concrete background, will not the later work with algebraic polynomials take on a new meaning and escape more algebraic symbolism and the national committee's criticism of mere mechanical routine?

The third simple instrument which I suggest in the construction of formulas is squared paper. Area and volume formulas are much simplified by its use. We have found that this quarter-inch squared paper is inexpensive and very satisfactory. For example, by drawing a parallelogram on this squared paper, drawing perpendicular lines from the upper vertices to the base, and base extended, counting the number of squares in each right triangle thus formed, and adding first one right triangle to the central portion and then the other, it can be shown that the area of a parallelogram is equal to the area of a rectangle in much the same way as is done in the formal proof in demonstrative geometry.[1]

[1] See page 80 of *General Mathematics*, Book One, by Schorling and Reeve.

I have found one sentence from the *Psychology of Secondary Education* by Judd helpful. Dr. Judd says:

"Encourage the pupils to go back to the fundamentals far enough so that they, at any time, will be able to rescue themselves from difficulties by building up thought processes which will carry them through the situation."

In developing the formula for the area of a trapezoid, I have kept this sentence in mind, so that the first time it was taken and whenever after that the formula came up, the pupils drew the diagonal and expressed the area of the trapezoid in terms of the areas of the two triangles formed. On the January examination for New York State in General Mathematics *B* there was a question which required the use of the formula for the area of a trapezoid. I was much concerned when I saw it because it was a formula that had not come up for a number of weeks preceding the examination. In fact, it really was not in the first term's work in the syllabus that we were using. However, when I came to examine the papers, I found that almost no one had missed this formula because they had gone back far enough to fundamentals and had expressed the formula in terms of the sum of the areas of the separate triangles. In this connection, the pupils were very much interested in generalizing. Using the formula for the area of the trapezoid and taking $b' = b$, the corresponding figure is the parallelogram and the formula is $A = \frac{1}{2}h (b + b)$, or $A = \frac{1}{2}h \times 2b$, or $A = bh$, which is the corresponding formula for the parallelogram. Again when $b' = b$ and when one of the angles is a right angle, we have the rectangle and the corresponding formula. When $b' = 0$, a triangle results and the formula becomes $A = \frac{1}{2}bh$, and once more, when $b' = b = h$ the formula becomes $A = \frac{1}{2}b(b + b)$, or $A = \frac{1}{2}b \times 2b$, and thus $A = b^2$, that is, the formula for the area of a square results.

The little boys of my study room were very busy the other day with paste, squared paper, and scissors, and they exhibited very proudly, the cubes which their algebra teacher had asked them to make. Not only is this work valuable for interest, but the seeing and handling of these models give a meaning to the corresponding volume formulas which could not be gotten in any other way.

To quote once more from the tentative syllabus in Junior High School Mathematics referred to above, this time on the subject

of "Variation" in the section on ninth grade algebra. "This work in variation should be done concretely by means of diagrams and models before a purely symbolic approach is attempted." I need hardly mention the possibilities with these models. If enough of the little equal cubes are put together, the dimensions of the original cube may be doubled or tripled and the relationship between the edges, the total surfaces and volumes studied.

Since I feel certain that everyone in this group uses the "Geometry Aid", the number line, in teaching the addition and subtraction of directed numbers, I will pause on this topic only long enough to suggest the use of angles as another teaching device. Those pupils who have learned already to use the protractor and have added and subtracted angles will find this another interesting means of drill. If it is understood that an angle formed by the rotation of a line counter-clockwise is positive, and one clockwise is negative, then such problems as the following can be done. Find the final position of a line, which, starting from the horizontal position OX, swings successively through the following rotations: $+72°, -38°, -10°, +25°$. Or the problem may be one of subtraction by the additive method. Through how many degrees must the line OR' turn to reach the position OR if the angle $XOR' = -10°$, and the angle $XOR = 25°$? Following the directions of the angles both with his hand and his eye, the pupil certainly knows what he is doing. He can use the protractor to check each step of his work, so that if he makes an error, he is able to discover just where it is by means of this check.

If one who has used the number line in teaching addition and subtraction of positive and negative numbers wishes to be consistent in his teaching, it may be used again as an explanation for multiplication. The geometric interpretation of this, for example; $(+4)(+2)$ would mean that a line segment $+4$ units is laid off twice, to the right of 0, that is in its own direction, and $(-4)(-2)$ would mean a line segment $+4$ units long laid off twice in the opposite direction, that is to the left of 0, and finally $-(-4)(-2)$, a line segment -4 units in length is laid off twice in the opposite direction or to the right of 0. This is not, possibly, the best way, but I am considering it, on the basis of consistency only.

We all of us need to do more or less work with arithmetical fractions as an introduction to the corresponding algebraic work. Here again squared paper aids in visualizing and understanding. For example, if a rectangle six units long is divided first into halves, then each half into thirds, one of these thirds shaded will represent one third times one half or one sixth of the whole. Likewise, one half times one third can be illustrated by dividing a given rectangle first into three equal parts, then each third into halves. If three fourths of a rectangle is shaded, then two thirds of this shaded portion is outlined. This outlined part is shown to contain six twelfths or one half of the whole, and all of this pictures the product of two thirds and three fourths. Cases of simple division as two thirds divided by two can be shown in a similar way.

I have drawn on cards some of the figures to be found in recent text-books of algebra to be used as teaching devices in multiplication, squaring a binomial, and completing the square. I will say a few words in order to point out what I feel are advantages to be gained by their use. The expression $5(3a+2)$ means the sum of the areas of two rectangles. Pupils with previous training in adding line segments have no difficulty in representing $3a+2$ either on squared paper or on plain paper with the use of compasses, and after completing the rectangle with an altitude of 5 units find the $15a$ and 10 as the areas of the separate rectangles. If enough of this work is done in the beginning when multiplying binomials or trinomials by monomials, the pupils rarely make the mistake of not multiplying each term by the common factor because they have a picture as a background. One book uses this pictorial device to teach division with a monomial divisor. The problem is stated thus: "Given the area of a rectangle $6x^2+4xy+8yz$ and altitude $2x$, find the base of the rectangle." The removal of a common monomial factor can be shown by letting the given expression represent the area of a rectangle. The pupils who have had this other training know that the given rectangle can be divided into as many rectangles as there are terms in the polynomial, that these rectangles are adjacent and have a common altitude. The problem becomes one of finding the common altitude and the bases of these separate rectangles. After this first work has given the pupils a mental image this type of factoring is found to be easy and is liked.

By means of the diagram for the square of a binomial, the pupils see that the square consists of two squares and two rectangles, and are saved the error of forgetting the middle term in the expansion of $(a + b)^2$. Again the diagram used to complete the square answers the questions "why half the coefficient of x," and "why the square of half the coefficient of x," in a way which no amount of explaining can do. I used this diagram in a Senior Review class in intermediate algebra and found that they were very much interested in seeing that it worked in all cases. There was no end to the different forms which they suggested for trial, and I feel that they all have a different understanding and clearer conception of completing the square.

Dr. Thorndike says in *The Psychology of Algebra:* "To make problem solving in school more like problem solving in business, industry and science, problems lacking essential details are important."

Dr. Vera Sanford, in her article "Extraneous Details" in the last MATHEMATICS TEACHER says that text-book problems that lack essential details are even more interesting, but they are fewer in number.

I checked the last statement when reviewing some of the more recent text-books of elementary algebra last winter and read many problems with the purpose of finding those lacking essential details. I found that they were very few in number in the books.

What is the best source of such problems? I think those that use some geometrical principles, for example: the angle sum of the triangle where the pupil needs to know that the sum of the interior angles of a triangle equals a straight angle. Is not this problem as good for drill in the solution of the equation? Is it not as real and interesting to the pupils as some other that it might replace?

Perimeter problems can be done both algebraically and geometrically, the geometric solution, following and checking each step of the algebraic work, for example: if the perimeter of an isosceles triangle and the relation between the base and the side is given, both base and side can be found and this work can be checked by the corresponding geometric construction. Of course, here the fact of the equality of the two sides must be

known, but that is certainly a fact that anyone ought to know whether he is going on into demonstrative geometry or not. Computing the sides of the right triangle using the corresponding quadratic equations and the possibilities of construction on squared paper, I can just mention. A simple diagram which may be no more than a straight line on which he marks the starting point and the directions is an important aid, I think, in the understanding of time, rate, and distance problems. If this work is done to scale, or is drawn on squared paper, it gives a splendid check on a solution.

I could go on indefinitely, of course, on problem work, but I want you to have an opportunity to give some of the suggestions and additions which have come to your mind during this time, and with just one quotation from one of the little boys of the Freshman class, I will bring this to a close.

To appreciate this you need to know that the boy was a failure last term and in consequence was put into a double algebra class where they were doing a great deal of this work. He came down the other day from his algebra class all excited, even taking a few minutes from his lunch hour saying:

"Miss Wannenmacher, algebra grows more and more interesting every day."

I believe in this work, not only for slow pupils, but for all pupils.

DETROIT MATHEMATICS CLUB A BRANCH OF THE NATIONAL COUNCIL

At its last meeting on December 6th the Detroit Mathematics Club voted unanimously to become a branch of the National Council. They have selected Mr. Everitt Corrigan of the Jefferson Intermediate School to act as "Detroit Representative" of the National Council. It is the business of such club representatives to assist the MATHEMATICS TEACHER in securing good material for publication, to get more members for the Council, and in general to help the Council in any way they can. It is to be hoped that every branch in the country will select some person to act as its representative.

NEWS NOTES

The Mathematics Section of the Western District of the New York State Teachers Association held its fall meeting in the Auditorium of School Number 3 at Buffalo on Friday afternoon, November 2d. Professor Joseph F. Philiippi, of the State Teachers College of Buffalo, presided. Professor Reeve, of Teachers College, discussed "The New Course in the Junior High School," and Mr. Seymour, State Supervisor of Mathematics in New York, discussed "The New State Syllabi in Mathematics."

The Mathematics Section of the Minnesota Educational Association held its annual meeting in Minneapolis on November 9th. C. N. Stokes, head of the Mathematics Department of the University of Minnesota High School, presided. Twelve hundred teachers of mathematics and others were in attendance. This comes close to being a record if it is not one.

Dr. F. B. Knight, of the State University of Iowa, delivered the first address, on "Modern Trends in the Teaching of Arithmetic."

The second address was delivered by Dr. W. D. Reeve of Teachers College. His topic was "Significant Tendencies in the Teaching of Secondary Mathematics."

The third speaker was Dr. Boyd H. Bode of Ohio State University. His topic was "The Teaching of Mathematics." He pleased the audience when he expressed himself as being a general educator in harmony with "Mathematics as a Mode of Thinking" and with "Mathematics as a Means of Adaptation."

The regular meeting was followed by a dinner meeting in the Sun Room of the Curtis Hotel sponsored by the Mathematics Clubs of both Minneapolis and St. Paul. Dr. Knight and Dr. Reeve spoke at this meeting. There were 106 people present at the dinner.

ONE HUNDRED PERCENT MEMBERSHIP!

The mathematics departments in the following schools have reported 100 percent membership in the National Council of Teachers of Mathematics:

1. Oak Park Hill School, Oak Park, Ill.
2. Jefferson Intermediate School, Detroit, Mich.
3. Bryant High School, New York City.
4. Jamaica High School, Jamaica, L. I.
5. Lafayette High School, Buffalo, N. Y.

This movement on the part of the various schools to get each teacher to become a member of the Council is a laudable one. It is only in this way that the work of the Council can be properly extended. The MATHEMATICS TEACHER will be glad to have such reports from other schools.

NEW BOOKS

Mathematische Quellenbücher. I. Rechnen und Algebra. HEINRICH WIELEITNER. Berlin, Otto Salle, 1927. Pp. viii + 75. Price 1.40 RM.

Mathematische Quellenbücher. II Geometrie und Trigonometrie. *Ibid.*, 1927. Pp. viii + 68. Price 1.40 RM.

Mathematische Quellenbücher. III. Analytische und Synthetische Geometrie. HEINRICH WIELEITNER. *Ibid.*, 1928. Pp. viii + 89. Price 2.50 RM.

Apollonius. FRITZ KLIEM. *Ibid.*, 1927. Pp. [viii] + 75. Price 2.40 RM.

Die Hauptfragen der heutigen Naturphilosophie. B. BAVINK. *Ibid.*, 1928. Pp. viii + 121. Price 3.30 RM.

The above mentioned works have recently appeared in the German series, *Mathematisch-Naturwissenschaftlich-Technische Bücherei*, edited by Dr. E. Wasserloos of Essen and Dr. Georg Wolff of Hannover. Dr. Wolff may be personally known to some of our readers, he having recently visited this country and having written a number of articles concerning our secondary schools. The series is one of several of the same general type, being intended primarily for teachers, but at the same time being admirably adapted to the purposes of school libraries. To American teachers who may be familiar with this series and others of the same nature it must always be a matter of regret that we cannot, in this country, publish scholarly but inexpensive works of this kind. They would assist greatly in raising the standard of scholarship in our schools and of adding interest to the instruction which we give.

To teachers of mathematics the first three of these books will appeal very strongly. They are the *Quellenbücher* (Source Books) of the work in the first fourteen years of American school life. Dr. Wieleitner is well known as one of the leading historians of mathematics of our time, and as a teacher of recognized ability. In his *Rechnen und Algebra* he gives German translations of a considerable number of the significant sources of the subjects taught in arithmetic and algebra. For example, he has two pages upon the late medieval Roman numerals, with extracts from the recently-published "Traicté d'arismetique pour pratique par gectoners;"[1] from Böschensteyn's (1514) treat-

[1] A work made known by Professor Lynn Thorndike in the *American Math. Monthly*, XXXIII, 24.

ment of the Regeldetri (Rule of Three); from the Rhind Papyrus (Ahmes, *c.* 1600 B.C.)—a new and very elaborate edition of which, by Dr. A. B. Chace, is just coming from the press in this country—the selection relating to fractions; from Euclid on geometric series; and from Nicomachus on proportion, the selection being made from another American work.[2] There are also extracts from Diophantus, al-Khowârizmî,[3] Adam Riese,[4] Rudolff (1525), Grammateus (Schreiber, *c.* 1518), Stifel (early approach to logarithms, 1544), Cardan (early approach to imaginaries, 1545), Bombelli (1572), Vieta (1593), Stevin (decimals, 1585), and Descartes[5] (1637).

In his second "Quellenbuch" (Geometry and trigonometry) Dr. Wieleitner gives twenty-one sources, as follows: the lunes of Hippocrates, from the work of Simplicius (6th cent. A.D.); the geometric proofs of certain algebraic identities, from Euclid; the intersecting chords, from Euclid; the intersection of great circles on a sphere, from Theodosius; the Pappus proof of the generalized Pythagorean; the Ptolemaic theorem on the inscribed quadrilateral; the addition theorem for cosines, from Ptolemy; the volume of the frustum of a pyramid, from Heron; the circumference and area of a circle, by a German surveyor of *c.* 1400;[6] the spherical cosine-theorem, from Regiomontanus (1476, published in 1533); the approximate construction of the regular pentagon, from an anonymous work of 1484; Heron's formula for the area of a triangle (possibly known to Archimedes); the names of the sides of a right triangle (*ypotenusa, basis, cathetus*), from Rudolff (1525); the construction of a third proportional, from the *Vnderweysung der messung* . . . by the artist Albrecht Dürer (1525); measurements of heights with the quadrant or astrolabe, from Stöffler (1536); the spherical sine-theorem from Copernicus (1543); the trigonometric computation of the angles of a polygon, from Giabattista Benedetti

[2] The translation by Professor M. L. D'Ooge, with commentary by Professors Robbins and Karpinski.

[3] Again with credit to an American edition, that of Professor Karpinski (New York, 1915).

[4] Particularly interesting as being from an unpublished manuscript now in Marienberg i. S., written in 1524.

[5] Also with reference to the American translation, Chicago, 1925. Early step toward the fundamental theorem of algebra.

[6] The so-called *Geometria Culmensis* of Conrad of Jungingen, 1393–1407, the oldest geometric MS. in German.

(1585); the tangent law, from Fincke (1583); Vieta's treatment of oblique triangles (1593); the later treatment of the addition theorem for sines, from Maier (1729); and De Moivre's theorem on imaginaries as set forth by Euler (1748).

Dr. Wieleitner's third work (*Analytische und Synthetische Geometrie*) contains eighteen extracts. These include selections from Apollonius on conic sections; the introduction of coordinates by Fermat (*c.* 1636, thus antedating Descartes's publication) and by Descartes (1637); De Beaune, van Schooten, l'Hospital, and Euler on conics; Desargues on involution; Pascal on the inscribed hexagram; Poncelet on projective geometry; Möbius on anharmonic ratios; and Steiner on the relation of conics to projective geometry.

No source book was ever written, or ever will be, that satisfies all readers. It would be very easy to suggest other sources which we would like to see in books like those of Dr. Wieleitner, but it would not be easy to suggest any of his sources that we would like to have omitted. Looking at the books in this light, they may be said to serve their purpose well. They will be of great help to any teacher of elementary mathematics, including elementary analytic geometry, who reads them. They should have place in all mathematical libraries.

Dr. Kliem is known to German scholars, and to students of the history of mathematics generally, for his excellent translation (1914) of Sir Thomas Heath's *Archimedes* (Cambridge, 1897). Because of this work he was well prepared to undertake such a brief presentation as the present one on the essential features of the *Conics*. What modern students especially need is not so much a critical examination of every proposition of Apollonius as a clear understanding of his method. To see what was accomplished in conic sections by the Euclidean type of proof, without the help of modern analysis, is to understand and appreciate more fully the remarkable intellectual powers of the Greeks. For this purpose the work by Dr. Kliem will be very helpful. He has selected well and he has called attention to the distinction mentioned—that between the Greek method and the one which modern analysis has given to the world.

The work of Professor Bavink is unfortunately not so directly connected with that of the American teacher of mathematics as with that of the German teacher, who so commonly combines

mathematics and physics. It is, however, none the less important. It sets forth the leading questions in contemporary natural philosophy, and cannot fail to be inspiring to the teacher of physics and helpful to the beginner in his studies. In the first volume (the only one that has yet reached us) the author approaches the problem from the philosophic standpoint, considering the distinction between the natural philosophy of earlier times and the physics of today, together with the problem of what constitutes knowledge, and a discussion of the meaning of truth, of intuitive belief, of idealism, and of positivism, pragmatism, conventionalism, and critical realism. The purpose of this philosophical approach is to establish a basis for the discussion, in two chapters, of (1) Space, time, and motion; and (2) The process of arriving at knowledge in the domain of physics. While such a treatment may seem to most teachers as too abstract for their purposes, it is just this kind of approach that gives one a feeling of equipment for a clear presentation of the postulates from which physical theories must be deduced.

David Eugene Smith

Exercises and Tests in Algebra Through Quadratics. By David Eugene Smith, William David Reeve, and Edward Longworth Morss. Ginn and Company, 1926. Price: $0.60.

This booklet of 224 sets of exercises provides material which may be used either to develop algebraic technique, or to test the mastery of this technique. It is evident that the purpose of the authors was not to provide drill solely for its own sake, but first to facilitate the acquisition of important skills by the student to the end that he may have more time to devote to other parts of mathematics, and secondly to provide the teacher with carefully graded material that may be used for diagnosis, driil, and testing whether by a class or by individuals and which will be economical of time both in administering and in scoring. The exercises are arranged in pairs, the even numbered tests being parallel to the ones that preceed them. Thus, if a preliminary diagnosis shows a particular weakness, a test is at hand covering the same topics for use after a number of practice periods. The record blank at the back of the booklet enables the student to keep note of his progress.

Besides the usefulness for practice work, this booklet will be

appreciated for its value in testing algebraic technique. To take a specific case, suppose a class is to be tested in the use of signed numbers. It is possible to make a selection from fourteen pages of exercises on that subject from which comprehensive tests may be selected. One such test, for example, includes all possible combinations of signs in addition. Students prefer these clearly printed pages to mimeographed ones or to questions written on the blackboard. Teachers appreciate the convenience of pages that may be removed from the book for easy handling and they also appreciate the ease with which these tests may be scored.

Time limits appear on each test. These should be considered as being suggestive rather than obligatory. They have value as a spur to an indolent class, but the reviewer's experience with them indicates that one of their greatest values is in convincing a class that an assignment is not an unreasonable one.

It is evident that no class will ever attempt all of the tests. The booklet is rich both in the variety of topics and in the careful analysis of the work under each topic. The treatment of the simple equation is particularly fine.

Certain of the tests are devoted to the study of problems, grouped according to their subject matter. This may be criticized as having less intrinsic value than the analysis of miscellaneous problems, but such a comment would be answered by a study of the topics that are involved. Surely a page devoted to the rate-time-distance formula is none too much if a student has difficulty in comprehending it.

Besides being an answer key, the *Teacher's Manual* gives a brief statement of the purpose of each test and these statements should be a valuable aid in the selection of tests suitable for particular situations.

On the whole, the booklet is notable for its assistance in making work in algebraic technique more effective and thus freeing student and teacher for those pursuits in mathematics which cannot be following unless the technique is at hand, and which to many people constitute the deeper values of the course.

VERA SANFORD

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